The Projective Theory of Ruled Surfaces

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Abstract: The aim of this paper is to get some results about ruled surfaces which configure a projective theory of scrolls and ruled surfaces. Our ideas follow the viewpoint of Corrado Segre, but we employ the contemporaneous language of locally free sheaves. The results complete the exposition given by R. Hartshorne and they have not appeared before in the contemporaneous literature.

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Introduction: Through this paper, a *geometrically ruled surface*, or simply a *ruled surface*, will be a $\mathbf{P}^1$-bundle over a smooth curve X of genus g. It will be denoted by $\pi : S = \mathbf{P}(\mathcal{E}_0) \longrightarrow X$ and we will follow the notation and terminology of R. Hartshorne's book [8], V, section 2. We will suppose that $\mathcal{E}_0$ is a normalized sheaf and X_0 is the section of minimum self-intersection that corresponds to the surjection $\mathcal{E}_0 \longrightarrow \mathcal{O}_X(\mathfrak{e}) \longrightarrow 0$, $\bigwedge^2 \mathcal{E} \cong \mathcal{O}_X(\mathfrak{e})$. Which are the linear equivalence classes $D \sim mX_0 + \mathfrak{b}f$, $\mathfrak{b} \in Pic(X)$, that correspond to very ample divisors?. When $g = 1$ and $m = 1$, a characterization is known ([8], V, ex.2.12), but the classification of elliptic scrolls obtained by Corrado Segre in [17] does not follow directly from this. A scroll is the birational image of a ruled surface $\pi : S = \mathbf{P}(\mathcal{E}_0) \longrightarrow X$ by an unisecant complete linear system.

The philosophy of this work is to develop a theory of ruled surfaces that allows their projective classification, by using the modern language of $\mathbf{P}^n$-bundles and rescuing the classical viewpoint introduced by C. Segre in [18]. This Segre's paper was reviewed with criticism by F. Severi in [19], but only some of the results of this work were reformulated nowadays. The study of directrix curves with minimum self-intersection and the formalization of the concept of ruled surface of general type was made by F. Ghione in [6]. The calculus of the genus of a curve on a ruled surface appeared in Ghione-Sacchiero [7]. The Hilbert scheme of the nonspecial ruled surfaces was studied by the second author in [1] and [16], where the property of maximal rank was proved. The theorem of

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C.Segre which says that $e \geq -g$ in a ruled surface $\pi : \mathbf{P}(\mathcal{E}_0) \longrightarrow X$ was proved by M. Nagata in [15], by H. Lange in [10], and it was generalized to higher rank by H. Lange in [11]. Finally, M. Maruyama studied ruled surfaces by using elementary transforms in [13]. He applies the classification theorem of Nagata (all geometrically ruled surface $\pi : \mathbf{P}(\mathcal{E}_0) \longrightarrow X$ is obtained from $X \times \mathbf{P}^1$ by applying a finite number of elementary transformations, [21], V,§1) to study the moduli of ruled surfaces of genus $g \leq 3$. Anyway, the results of this paper complete the exposition about ruled surfaces given in [8] and they have not appeared before in the contemporaneous literature.

The paper is organized in the following way:

§1: Ruled surfaces and scrolls.

§2: Unisecant linear systems on a ruled surface.

§3: Decomposable ruled surfaces.

§4: Elementary transformation of a ruled surface.

§5: Speciality of a scroll.

§6: Segre Theorems.

In §1 we introduce the basic facts about ruled surfaces and we relate them to the scroll. The classical authors define a scroll as a surface $R \subset \mathbf{P}^N$ such that there exists a line contained in R that passes through the generic point (see [20], 204). We show that any scroll is the birational image of a geometrically ruled surface $S = \mathbf{P}(\mathcal{E}_0)$ by an unisecant linear system. In a modern way, this is the equivalence between morphisms $\phi : X \longrightarrow G(1, N)$, where X is a smooth curve, and surjections $\mathcal{O}_G^{N+1} \longrightarrow \mathcal{E}$, where $\mathcal{E} = \phi^* U$ is the locally free sheaf of rank 2 obtained from the universal bundle U.

In §2 we characterize when a complete linear system defined by an unisecant divisor $H \sim X_0 + \mathfrak{b}f$ in S is base-point-free. The most important result is Theorem 2.11 which describes the points where the regular map $\phi_H : S \longrightarrow \mathbf{P}^N$ is not a local isomorphism. Equivalently, this characterizes the singular locus of the scroll $R = \phi_H(S)$, $\phi_H^{-1}(sing(R)) = \{x \in S/x$ is a base point of $|H - Pf|, P \in X\}$, and when $|H|$ is very ample.

In §3 we consider a decomposable ruled surface $\mathcal{E}_0 \cong \mathcal{O}_X \oplus \mathcal{O}_X(\mathfrak{e})$. There exist two disjoint sections X_0 and X_1, which correspond to the surjections $\mathcal{E}_0 \longrightarrow \mathcal{O}_X(\mathfrak{e}) \longrightarrow 0$ and $\mathcal{E}_0 \longrightarrow \mathcal{O}_X \longrightarrow 0$. We prove some results that localize the base points of a unisecant complete linear system $H \sim X_0 + \mathfrak{b}f$ over X_0 or X_1. We study the existence of sections in $|H|$ and we give a sufficient condition for ϕ_H to be an isomorphism in points out of X_0 or X_1. The main result of this section is Theorem 3.10, where we describe the support of the singular locus of the regular map $\phi_H : S \longrightarrow R \subset \mathbf{P}^N$. We finish this section studying the base-point-free and very ample m-secant complete linear systems.

In §4 we make a classical study of the elementary transformation of a ruled surface. We describe some elementary properties and we show that the elementary transform corresponds to the projection of a scroll from a nonsingular point.

The study of how the divisor $\mathfrak{e}$ is transformed by the elementary transformation at a point x in the minimum self-intersection section allow us to give an easy demonstration of the result of C.Segre (Corollary 4.10): any indecomposable scroll is obtained from a decomposable one by applying a finite number of elementary transformations. We use that $e = -\partial(\mathfrak{e}) \leq 2g-2$ in a decomposable ruled surface.

The main result of this section is Theorem 4.12, where we identify the elementary transforms of a decomposable ruled surface at a point x according to its position.

In §5, we introduce the special ruled surfaces. Then we use the elementary transformation to give a geometrical meaning, according to Riemann-Roch, of the speciality of a scroll. In this way, we pose the problem of the existence of scrolls with speciality 1 over a smooth curve of genus $g \geq 1$ and such that any special scroll is obtained by projection from them. This problem is solved in [5].

Finally, in §6, we rescue the results of Segre in [18] about special ruled surfaces. We conserve the spirit of Segre's methods, although we write them in modern way. In fact, Segre proved that a special ruled surface of genus g and degree $d \geq 4g-2$ always has a special directrix curve, but the condition over the degree is not necessary: any special ruled surface has a special directrix curve (see [5]).

Most of the results that appear in this paper generalize to higher rank and will be studied in a forthcoming paper.

1 Ruled surfaces and scrolls.

Definition 1.1 *A geometrically ruled surface, or simply ruled surface, is a surface S, together with a surjective morphism $\pi : S \longrightarrow X$ to a smooth curve X, such that the fibre S_x is isomorphic to $\mathbf{P}^1$ for every point $x \in C$, and such that π admits a section (i.e., a morphism $i : X \longrightarrow S$ such that $\pi \circ i = id_X$).*

Proposition 1.2 *If $\pi : S \longrightarrow X$ is a ruled surface, then there exist a locally free sheaf $\mathcal{E}$ of rank 2 on X such that $S \cong \mathbf{P}(\mathcal{E})$ over X. Conversely, every such $\mathbf{P}(\mathcal{E})$ is a ruled surface over X. If $\mathcal{E}$ and $\mathcal{E}'$ are two locally free sheaves of rank 2 on X, then $\mathbf{P}(\mathcal{E})$ and $\mathbf{P}(\mathcal{E}')$ are isomorphic as ruled surfaces over X if and only if there is an invertible sheaf $\mathcal{L}$ on X such that $\mathcal{E}' \cong \mathcal{E} \otimes \mathcal{L}$.*

Proof: See [8], V, 2.2. ∎

If $\pi : S \longrightarrow X$ is a ruled surface, we can choose $S \cong \mathbf{P}(\mathcal{E}_0)$ where $\mathcal{E}_0$ is a locally free sheaf of rank 2 on X with the property $H^0(\mathcal{E}_0) \neq 0$ but for all invertible sheaves $\mathcal{L}$ on X with $deg(\mathcal{L}) < 0$, we have $H^0(\mathcal{E}_0 \otimes \mathcal{L}) = 0$. In this case we say $\mathcal{E}_0$ is *normalized*. The sheaf $\mathcal{E}_0$ is not determined uniquely, but it is determined $e = -deg(\mathcal{E})$.

Let $\mathfrak{e}$ be the divisor on X corresponding to the invertible sheaf $\bigwedge^2 \mathcal{E}_0$, then $e = -deg(\mathfrak{e})$. Moreover, there is a section $i : X \longrightarrow S$ with image X_0, such that $\mathcal{O}_S(X_0) \cong \mathcal{O}_S(1)$.

Proposition 1.3 *Under the above assumptions:*

$$Pic(S) \cong \mathbf{Z} \oplus \pi^* Pic(X)$$

where $\mathbf{Z}$ *is generated by* X_0*. Also*

$$Num(S) \cong \mathbf{Z} \oplus \mathbf{Z}$$

generated by X_0 *and* f*, and satisfying* $X_0.f = 1$*,* $f^2 = 0$*.*

Proof: See [8], V, 2.3. ∎

Thus, if $\mathfrak{b} \in Div(X)$, we denote the divisor $\pi^*\mathfrak{b}$ on S by $\mathfrak{b}f$. Therefore, any element of $Pic(S)$ can be written $nX_0 + \mathfrak{b}f$ with $n \in \mathbf{Z}$ and $\mathfrak{b} \in Pic(X)$. Any element of $Num(X)$ can be written $nX_0 + bf$ with $n, b \in \mathbf{Z}$. A linear system $|nX_0 + \mathfrak{b}f|$ will be called *n-secant* because it meets each generator at n points.

Proposition 1.4 *Let* $\mathcal{E}$ *be a locally free sheaf of rank* 2 *on the curve* X*, and let* S *be the ruled surface* $\mathbf{P}(\mathcal{E})$*. Let* $\mathcal{O}_S(1)$ *be the invertible sheaf* $\mathcal{O}_{\mathbf{P}(\mathcal{E})}(1)$*. Then there is a one-to-one correspondence between sections* $i : X \longrightarrow S$ *and surjections* $\mathcal{E} \longrightarrow \mathcal{L} \longrightarrow 0$*, where* $\mathcal{L}$ *is an invertible sheaf on* X*, given by* $i^*\mathcal{O}_S(1)$*.*

Furthermore, if D *is a section of* S*, corresponding to the surjection* $\mathcal{E} \to \mathcal{L} \to 0$*, and* $\mathcal{L} = \mathcal{O}_X(\mathfrak{a})$ *for any divisor* $\mathfrak{a}$ *on* X*, then* $deg(\mathfrak{a}) = X_0.D$*, and* $D \sim X_0 + (\mathfrak{a} - \mathfrak{e})f$*.*

Proof: See [8], V, 2.6 and 2.9. ∎

If $\mathcal{E}_0$ is a normalized sheaf and X_0 the corresponding section of the ruled surface $\pi : S \longrightarrow X$, we have that:

$$\pi_*\mathcal{O}_S(X_0) \cong \mathcal{E}_0$$

Moreover, if $H \sim X_0 + \mathfrak{b}f$, by the projection formula:

$$\pi_*\mathcal{O}_S(H) \cong \pi_*(\mathcal{O}_S(X_0) \otimes \pi^*\mathfrak{b}) \cong \mathcal{E}_0 \otimes \mathcal{O}_X(\mathfrak{b})$$

Since $R^i\pi_*\mathcal{O}_S(H) = 0$ for any $i > 0$, we have that $H^i(\mathcal{O}_S(H)) = H^i(\mathcal{E}_0 \otimes \mathcal{O}_X(\mathfrak{b}))$.

From this and from the definition of normalized sheaf, we see that the curve X_0 is the minimum self-intersection curve on S and $X_0^2 = -e$.

The image of a ruled surface by the map defined by an unisecant base-point-free linear system is a surface containing a onedimensional family of lines.

Definition 1.5 *A scroll $R \subset \mathbf{P}^N$ is an algebraic surface such that it has a line passing through the generic point. The lines of the scroll are called generators.*

Let $R \subset \mathbf{P}^N$ be a scroll. Let $\overline{H}$ be a generic hyperplane section of R. $\overline{H}$ is smooth away from the singular locus of R. Thus there is an open set $U \subset \overline{H}$, such that there is a unique generator passing through any point.

Let $G(1,N)$ be the Grassmaniann parameterizing the lines of $\mathbf{P}^N$. We have a map:

$$U \longrightarrow G(1,N)$$

which applies each point of U over the unique generator passing through it.

The map extends uniquely to the nonsingular model X of $\overline{H}$:

$$\eta : X \longrightarrow G(1,N)$$

If X is a curve of genus g, we say that R has genus g, that is, we define the genus of R as the geometric genus of the generic hyperplane section.

Definition 1.6 *Let $V \subset \mathbf{P}^N$ be a projective variety in $\mathbf{P}^N$. We say that V is projectively normal, when there is not any variety $V' \subset \mathbf{P}^{N'}$, with $N' > N$ and $deg(V) = deg(V')$ such that V' projects over V.*

Proposition 1.7 *A linearly normal scroll R is the image of a unique ruled surface S by the birational map defined by a base-point-free unisecant complete linear system $|H|$.*

Proof: Let $R \subset P^N$. Consider the corresponding map $\eta : X \longrightarrow G(1,N)$. We build the following incidence variety:

$$\begin{tikzcd}[column sep=small] & & \mathbf{P}^N \\ G(1,N)\times\mathbf{P}^N \leftarrow X\times\mathbf{P}^N\supset\mathcal{J}_X := \{(P,x)/x \in l_{\eta(P)}\} \arrow[urr, "q"] \arrow[drr, "p"'] & & \\ & & X \to G(1,N) \end{tikzcd}$$

$\mathcal{J}_X$ and X are smooth varieties and the map $p : \mathcal{J}_X \longrightarrow X$ has fibre $\mathbf{P}^1$ and surjective differential. Then, applying Enriqües–Noether Theorem (see [2], II), there exists an open set $U' \subset X$ verifying $p^{-1}(U') \simeq U' \times \mathbf{P}^1$. Since X is a smooth curve, we deduce that $p : \mathcal{J}_X \longrightarrow X$ has a section and it is a geometrically ruled surface.

The image of the projection q is exactly the scroll R on $\mathbf{P}^N$. The generic fibre of q is a point. Consider the invertible sheaf $\mathcal{L} \cong q^*\mathcal{O}_{P^N}(1)$. Their global sections correspond to the complete linear system $|H|$,where $H := q^*\overline{H}$. It is an unisecant linear system, because it meets the generic generator at a unique point. The map q is determined by a linear subsystem $\delta \subset |H|$, so $|H|$ is base-point-free.

But R is linearly normal, so $H^0(\mathcal{O}_{\mathcal{J}_X}(H)) \cong H^0(\mathcal{O}_{P^N}(1))$. From this q is determined by the complete linear system.

Note that the construction does not depend of the election of the hyperplane section, because any two hyperplane sections are birational equivalent. In fact, the ruled surface J_X is unique:

If we suppose R defined by the birational map determined by a base-point-free unisecant linear system $|H'|$ over the ruled surface $\pi : \mathbf{P}(\mathcal{E}) \longrightarrow X$:

$$\phi_{H'} : \mathbf{P}(\mathcal{E}) \longrightarrow R \subset \mathbf{P}^N$$

we can define a birational map $\eta' : X \longrightarrow G(1, N)$ which applies a point $P \in X$ on the line $\phi_{H'}(Pf)$ on $\mathbf{P}^N$. The maps η and η' are equal up to automorphism of X and then the incidence variety $\mathcal{J}_X$ is isomorphic to $\mathbf{P}(\mathcal{E})$. ■

Definition 1.8 *Let $R \subset \mathbf{P}^N$ be a linearly normal scroll, let S be a ruled surface and let $|H|$ be a base-point-free unisecant linear system defining a birational map $\phi_H : S \longrightarrow \mathbf{P}^N$. If $\phi_H(S) = R$, then we say that S and H are the ruled surface and the linear system associated to R.*

Definition 1.9 *A directrix curve of a scroll is a curve meeting each generator at a unique point.*

The directrix curves of a scroll R correspond to the sections of the associated ruled surface S. Suppose that R is the image of S by the map defined by the linear system $|X_0 + \mathfrak{b}f|$. We will denote the image of a section D of S by $\overline{D} \subset R$. The curve has degree $deg(\overline{D}) = D.X_0 + deg(\mathfrak{b})$. The degree of the scroll R is $(X_0 + \mathfrak{b}f)^2 = X_0^2 + 2deg(\mathfrak{b})$.

The minimum self-intersection curve X_0 of S corresponds to the minimum degree directrix curve of R. If we take two sections $D_1 \sim X_0 + \mathfrak{a}_1 f$ and $D_2 \sim$

$X_0 + \mathfrak{a}_2 f$ on S, they have non negative intersection. Thus:

$$\begin{array}{rl} deg(\overline{D_1}) + deg(\overline{D_2}) & = 2X_0^2 + 2deg(\mathfrak{b}) + deg(\mathfrak{a}_1) + deg(\mathfrak{a}_2) = \\ & = X_0^2 + 2deg(\mathfrak{b}) + D_1.D_2 \geq deg(R) \end{array}$$

We see that the sum of the degree of two directrix curves of R is greater than or equal to the degree of R.

We have seen that the study of the scrolls is equivalent to the study of geometrically ruled surfaces and their unisecant linear systems, but it is equivalent to the study of locally free sheaves of rank 2 over the base curve X too. In the next section we begin the study of the unisecant linear systems on a ruled surface and in this way we treat the study of the scrolls.

2 Unisecant linear systems on a ruled surface.

Let $\pi : S \longrightarrow X$ be a geometrically ruled surface. An unisecant complete linear system $|H| = |X_0 + \mathfrak{b}f|$ on S defines a rational map $\phi_H : S \longrightarrow \mathbf{P}^N$. The map ϕ_H is regular out of base points of $|H|$ and it is an isomorphism onto its image when $|H|$ is very ample. In this section we will study general conditions for an unisecant linear system to be base-point-free, to have irreducible elements and to define an isomorphism.

Lemma 2.1 *Let $\mathfrak{b}$ be a nonspecial divisor on X. Then, if $i \geq 0$:*

$$h^i(\mathcal{O}_S(X_0 + \mathfrak{b}f)) = h^i(\mathcal{O}_X(\mathfrak{b})) + h^i(\mathcal{O}_X(\mathfrak{b} + \mathfrak{e}))$$

Proof: Because S is a surface, it is sufficient to prove it for $i \leq 2$. Let us consider the exact sequence of X_0 on S:

$$0 \longrightarrow \mathcal{O}_S(-X_0) \longrightarrow \mathcal{O}_S \longrightarrow \mathcal{O}_{X_0} \longrightarrow 0$$

Tensoring with $\mathcal{O}_S(X_0 + \mathfrak{b}f)$, we get the cohomology sequence

$$\begin{array}{l} 0 \longrightarrow H^0(\mathcal{O}_S(\mathfrak{b}f)) \longrightarrow H^0(\mathcal{O}_S(X_0 + \mathfrak{b}f)) \longrightarrow H^0(\mathcal{O}_{X_0}(X_0 + \mathfrak{b}f)) \longrightarrow \\ \longrightarrow H^1(\mathcal{O}_S(\mathfrak{b}f)) \longrightarrow H^1(\mathcal{O}_S(X_0 + \mathfrak{b}f)) \longrightarrow H^1(\mathcal{O}_{X_0}(X_0 + \mathfrak{b}f)) \longrightarrow \\ \longrightarrow H^2(\mathcal{O}_S(\mathfrak{b}f)) \longrightarrow H^2(\mathcal{O}_S(X_0 + \mathfrak{b}f)) \longrightarrow H^2(\mathcal{O}_{X_0}(X_0 + \mathfrak{b}f)) \longrightarrow \end{array}$$

We have $h^i(\mathcal{O}_S(\mathfrak{b}f)) = h^i(\mathcal{O}_X(\mathfrak{b}))$ and $h^i(\mathcal{O}_{X_0}(X_0 + \mathfrak{b}f)) = h^i(\mathcal{O}_X(\mathfrak{b} + \mathfrak{e}))$. But $h^2(\mathcal{O}_X(\mathfrak{b} + \mathfrak{e})) = h^2(\mathcal{O}_X(\mathfrak{b})) = 0$. Since $\mathfrak{b}$ is nonspecial, $h^1(\mathcal{O}_S(\mathfrak{b}f)) = 0$ and the lemma follows. ■

Remark 2.2 Note that we have seen that the following inequality always holds:

$$h^i(\mathcal{O}_S(X_0 + \mathfrak{b}f)) \leq h^i(\mathcal{O}_X(\mathfrak{b})) + h^i(\mathcal{O}_X(\mathfrak{b} + \mathfrak{e})).$$

Furthermore, if we consider the linear system $|mX_0 + \mathfrak{b}f|$ with $m \geq 0$, for each $i > 0$ we have the exact sequence:

$$H^i(\mathcal{O}_S((m-1)X_0 + \mathfrak{b}f)) \longrightarrow H^i(\mathcal{O}_S(mX_0 + \mathfrak{b}f)) \longrightarrow H^i(\mathcal{O}_X(\mathfrak{b} + m\mathfrak{e}))$$

From this, we deduce that $h^i(\mathcal{O}_S((mX_0+\mathfrak{b}f)) \leq h^i(\mathcal{O}_X(\mathfrak{b}+m\mathfrak{e}))+h^i(\mathcal{O}_S((m-1)X_0 + \mathfrak{b}f))$. We continue in this fashion obtaining:

$$h^i(\mathcal{O}_S(mX_0 + \mathfrak{b}f)) \leq \sum_{k=0}^{m} h^i(\mathcal{O}_X(\mathfrak{b} + k\mathfrak{e}))$$

$\blacksquare$

Proposition 2.3 *Let S be a geometrically ruled surface and let $\mathfrak{b}$ be a divisor on X. Let $|H| = |X_0 + \mathfrak{b}f|$ be a complete linear system on S. Let P be a point in X. Then:*

1. *$|H|$ is base-point-free on the generator Pf if and only if $h^0(\mathcal{O}_S(H-Pf)) = h^0(\mathcal{O}_S(H)) - 2$.*
2. *$|H|$ has a unique base point on the generator Pf if and only if $h^0(\mathcal{O}_S(H-Pf)) = h^0(\mathcal{O}_S(H)) - 1$.*
3. *$|H|$ has Pf as a fixed component if and only if $h^0(\mathcal{O}_S(H - Pf)) = h^0(\mathcal{O}_S(H))$.*

Proof: Let us consider the trace of the linear system $|H|$ on the generator Pf:

$$0 \longrightarrow H^0(\mathcal{O}_S(H - Pf)) \longrightarrow H^0(\mathcal{O}_S(H)) \xrightarrow{\alpha} H^0(\mathcal{O}_{Pf}(H))$$

H meets each generator at a point, so $H^i(\mathcal{O}_{Pf}(H)) \cong H^i(\mathcal{O}_{\mathbf{P}^1}(1))$. Therefore $h^0(\mathcal{O}_{Pf}(H)) = 2$ and:

1. If $dim(\mathrm{Im}(\alpha)) = 2$, then the linear system $|H|$ traces on Pf the complete linear system of points of $\mathbf{P}^1$. Since this is base-point-free, $|H|$ is base-point-free on Pf.
2. If $dim(\mathrm{Im}(\alpha)) = 1$, then the linear system $|H|$ traces on Pf a unique point, so $|H|$ has a unique base point on the generator Pf.

3. If $dim(\mathrm{Im}(\alpha)) = 0$, then the generator Pf is a fixed component of the linear system $|H|$.

From the exact sequence we obtain $dim(\mathrm{Im}(\alpha)) = h^0(\mathcal{O}_S(H)) - h^0(\mathcal{O}_S(H - Pf))$, which completes the proof. ■

Corollary 2.4 *Let S be a geometrically ruled surface and $|H|$ an unisecant complete linear system on S. $|H|$is base-point-free if and only if for all $P \in X$, $h^0(\mathcal{O}_S(H - Pf)) = h^0(\mathcal{O}_S(H)) - 2$.*

Proposition 2.5 *Let $\mathfrak{b}$ be a divisor on X. If P is a base point of $|\mathfrak{b} + \mathfrak{e}|$, then $Pf \cap X_0$ is a base point of the complete linear system $|X_0 + \mathfrak{b}f|$.*

Proof: Let us study the trace of the linear system $|X_0 + \mathfrak{b}f|$ on X_0:

$$0 \to H^0(\mathcal{O}_S(\mathfrak{b}f)) \longrightarrow H^0(\mathcal{O}_S(X_0 + \mathfrak{b}f)) \longrightarrow H^0(\mathcal{O}_{X_0}(X_0 + \mathfrak{b}f)) \cong H^0(\mathcal{O}_X(\mathfrak{b} + \mathfrak{e}))$$

By hypothesis, P is a base point of $|\mathfrak{b} + \mathfrak{e}|$, so all divisors of $|X_0 + \mathfrak{b}f|$ trace on X_0 a divisor which contains P. We conclude that $Pf \cap X_0$ is a base point of $|X_0 + \mathfrak{b}f|$. ■

Lemma 2.6 *Let $\mathfrak{b}$ be a nonspecial divisor on X. Then:*

1. *If P is not a base point of $\mathfrak{b}$ and $\mathfrak{b} + \mathfrak{e}$, then the linear system $|X_0 + \mathfrak{b}f|$ has no base points on the generator Pf.*
2. *If P is a base point of $\mathfrak{b} + \mathfrak{e}$ but not of $\mathfrak{b}$, then the linear system $|X_0 + \mathfrak{b}f|$ has a unique base point on the generator Pf. This point is $X_0 \cap Pf$.*
3. *If P is a base point of $\mathfrak{b}$ but not of $\mathfrak{b} + \mathfrak{e}$, then the linear system $|X_0 + \mathfrak{b}f|$ has at most a base point on the generator Pf.*
4. *If P is a base point of $\mathfrak{b}$ and $\mathfrak{b} + \mathfrak{e}$, then the linear system $|X_0 + \mathfrak{b}f|$ has at least a base point on the generator Pf.*

Proof: By Proposition 2.3, it is sufficient to compute $h^0(\mathcal{O}_S(X_0 + \mathfrak{b}f))$ and $h^0(\mathcal{O}_S(X_0 + (\mathfrak{b} - P)f))$. Since $\mathfrak{b}$ is nonspecial, $h^0(\mathcal{O}_S(X_0 + \mathfrak{b}f)) = h^0(\mathcal{O}_X(\mathfrak{b})) + h^0(\mathcal{O}_X(\mathfrak{b} + \mathfrak{e}))$. We consider two cases:

1. If P is not a base point of $\mathfrak{b}$, $\mathfrak{b} - P$ is nonspecial because $\mathfrak{b}$ is nonspecial. Therefore, $h^0(\mathcal{O}_S(X_0 + (\mathfrak{b} - P)f)) = h^0(\mathcal{O}_X(\mathfrak{b} - P)) + h^0(\mathcal{O}_X(\mathfrak{b} + \mathfrak{e} - P)) = h^0(\mathcal{O}_X(\mathfrak{b})) - 1 + h^0(\mathcal{O}_X(\mathfrak{b} + \mathfrak{e} - P))$. Then, if P is not a base point of $\mathfrak{b} + \mathfrak{e}$, $h^0(\mathcal{O}_X(\mathfrak{b} + \mathfrak{e} - P)) = h^0(\mathcal{O}_X(\mathfrak{b} + \mathfrak{e})) - 1$ and the linear system is base-point-free on Pf. If P is a base point of $\mathfrak{b} + \mathfrak{e}$, then $h^0(\mathcal{O}_X(\mathfrak{b} + \mathfrak{e} - P)) = h^0(\mathcal{O}_X(\mathfrak{b} + \mathfrak{e}))$ and the linear system has a unique base point on Pf (by Proposition2.5, it is at $Pf \cap X_0$).

2. If P is base point of $\mathfrak{b}$, then $\mathfrak{b}-P$ is special. Then $h^0(\mathcal{O}_S(X_0+(\mathfrak{b}-P)f)) \leq h^0(\mathcal{O}_X(\mathfrak{b}-P)) + h^0(\mathcal{O}_X(\mathfrak{b}+\mathfrak{e}-P)) = h^0(\mathcal{O}_X(\mathfrak{b})) + h^0(\mathcal{O}_X(\mathfrak{b}+\mathfrak{e}-P))$. If P is not a base point of $\mathfrak{b}+\mathfrak{e}$, $h^0(\mathcal{O}_X(\mathfrak{b}+\mathfrak{e}-P)) = h^0(\mathcal{O}_X(\mathfrak{b}+\mathfrak{e}))-1$ and the linear system has at most a base point on Pf. If P is a base point of $\mathfrak{b}+\mathfrak{e}$, by Proposition 2.5, the linear system has at least a base point at $X_0 \cap Pf$.

Theorem 2.7 *Let S be a geometrically ruled surface and let $\mathfrak{b}$ be a divisor on X. There exists a section $D \sim X_0 + \mathfrak{b}f$ if and only if one of the following conditions holds:*

1. $h^0(\mathcal{O}_S(X_0+\mathfrak{b}f)) = 1$ *and* $h^0(\mathcal{O}_S(X_0+(\mathfrak{b}-P)f)) = 0$ *for all* $P \in X$.
2. $h^0(\mathcal{O}_S(X_0+\mathfrak{b}f)) > 1$, $h^0(\mathcal{O}_S(X_0+(\mathfrak{b}-P)f)) < h^0(\mathcal{O}_S(X_0+\mathfrak{b}f))$ *for all* $P \in X$ *and* $h^0(\mathcal{O}_S(X_0+(\mathfrak{b}-P)f) = h^0(\mathcal{O}_S(X_0+\mathfrak{b}f))-2$ *for the generic point* $P \in X$.

Proof: We first note that reducible elements of $|X_0+\mathfrak{b}f|$ contain at least one generator, so they are in linear subsystems $|X_0+(\mathfrak{b}-P)f|$.

Let us suppose $h^0(\mathcal{O}_S(X_0+\mathfrak{b}f)) = 1$. If $h^0(\mathcal{O}_S(X_0+(\mathfrak{b}-P)f)) = 1$ for some $P \in X$, then the unique effective divisor of $|X_0+\mathfrak{b}f|$ contains Pf so it is not irreducible. Conversely, if $h^0(\mathcal{O}_S(X_0+(\mathfrak{b}-P)f)) = 0$ for all $P \in X$, then the unique effective divisor of the linear system does not contain any generator, so it is irreducible.

Let us now suppose $h^0(\mathcal{O}_S(X_0+\mathfrak{b}f)) > 1$. If $h^0(X_0+(\mathfrak{b}-P)f) = h^0(\mathcal{O}_S(X_0+\mathfrak{b}f))$ for some $P \in X$. Then (by Lemma 2.3) Pf is a fixed component of the linear system, so there are not irreducible elements in $|X_0+\mathfrak{b}f|$.

If $h^0(X_0+(\mathfrak{b}-P)f) = h^0(\mathcal{O}_S(X_0+\mathfrak{b}f))-1$ for all $P \in X$, then (Proposition 2.3) the linear system $|X_0+\mathfrak{b}f|$ has a unique base point on each generator. Hence there exists a fixed unisecant curve in the linear system and, as $h^0(\mathcal{O}_S(X_0+\mathfrak{b}f)) > 1$ the generic element is not irreducible.

Conversely, if the codimension of linear subsystems $|X_0+(\mathfrak{b}-P)f|$ is 2 for the generic point and 1 for the remaining ones, then the reducible elements don't satisfy the linear system, so the generic element is irreducible. ∎

The curve X_0 is unique on its class of linear equivalence, except when the ruled surface is $X \times \mathbf{P}^1$.

Lemma 2.8 *Let $S = \mathbf{P}(\mathcal{E}_0) \longrightarrow X$ be a ruled surface. Then $h^0(\mathcal{O}_S(X_0)) = 2$ when $\mathbf{P}(\mathcal{E}_0) \cong \mathbf{P}^1 \times X$ and $h^0(\mathcal{O}_S(X_0)) = 1$ in other case.*

Proof: Since X_0 is the minimum self-intersection curve, it corresponds to the normalized sheaf $\mathcal{E}_0$. Then we have that $h^0(\mathcal{O}_S(X_0)) = h^0(\mathcal{E}_0) > 0$ and $h^0(\mathcal{O}_S(X_0 - Pf)) = h^0(\mathcal{E}_0 \otimes \mathcal{O}_X(-P)) = 0$, for any point $P \in X$. From this, $h^0(\mathcal{O}_S(X_0)) \leq h^0(\mathcal{O}_S(X_0 - Pf)) + 2 = 2$.

If $\mathbf{P}(\mathcal{E}_0) \cong \mathbf{P}^1 \times X$ then $\mathcal{E}_0 \cong \mathcal{O}_X \oplus \mathcal{O}_X$ and $h^0(\mathcal{O}_{P(\mathcal{E}_0)}(X_0)) = 2$.

Suppose that $h^0(\mathcal{O}_S(X_0)) = 2$. Then $|X_0|$ is a pencil of unisecant irreducible curves, because $h^0(\mathcal{O}_S(X_0 - Pf)) = 0$. If $X_0', X_0'' \in |X_0|$, $X_0'.X_0'' = -e$ must be positive, so $e \leq 0$. Suppose that $e < 0$. Then the curves X_0' and X_0'' have at least a common point. Because $|X_0|$ has dimension 1, the linear system has a base point. By Proposition 2.3, there is a point $P \in X$ such that $h^0(\mathcal{O}_S(X_0 - Pf)) > h^0(\mathcal{O}_S(X_0)) - 2 = 0$, but this is false. Therefore $e = 0$ and $\mathbf{P}(\mathcal{E}_0)$ has a pencil of disjoint unisecant curves. We have an isomorphism:

$$|X_0| \times X \xrightarrow{\cong} \mathbf{P}(\mathcal{E}_0)$$

that is, $\mathbf{P}(\mathcal{E}_0) \cong \mathbf{P}^1 \times X$. ■

Corollary 2.9 *Let $\mathfrak{b}$ and $\mathfrak{b} + \mathfrak{e}$ be effective divisors on X. If they have no common base points and $\mathfrak{b}$ is nonspecial, then there exists a section $D \sim X_0 + \mathfrak{b}f$. Furthermore if $\mathfrak{b}$ and $\mathfrak{b} + \mathfrak{e}$ are base-point-free, then the complete linear system $|X_0 + \mathfrak{b}f|$ is base-point-free.*

Proof: Because $\mathfrak{b}$ and $\mathfrak{b}+\mathfrak{e}$ are effective divisors, a generic point P is not a base point of both of them. By Propositions 2.3 and 2.6, $h^0(\mathcal{O}_S(X_0 + (\mathfrak{b} - P)f)) = h^0(\mathcal{O}_S(X_0 + \mathfrak{b}f)) - 2$.

If P is a base point of $\mathfrak{b}$ or $\mathfrak{b} + \mathfrak{e}$ (by hypothesis P is not a common base point), by applying Proposition 2.6, we obtain that $|X_0 + \mathfrak{b}f|$ has at most a base point on the generator Pf and $h^0(\mathcal{O}_S(X_0 + (\mathfrak{b} - P)f)) \leq h^0(\mathcal{O}_S(X_0 + \mathfrak{b}f)) - 1$.

Now, by applying Theorem 2.7, the first part of the statement follows.

According to Lemma 2.4, we see that if $\mathfrak{b}$ and $\mathfrak{b} + \mathfrak{e}$ are base-point-free, then the linear system $|X_0 + \mathfrak{b}f|$ is base-point-free. ■

Remark 2.10 A unisecant complete base-point-free linear system $|H|$ determines a morphism $\phi_H : S \longrightarrow \mathbf{P}^N$ that gives us a scroll $R = \phi_H(S)$ in $\mathbf{P}^N$.

The map is injective if it separates points, that is, given $x, y \in X$ with $x \neq y$, there is an element $D \in |H|$, such that $x \in D$, but $y \notin D$.

Furthermore, the differential is injective at $x \in S$ when it separates tangent vectors, that is, given $t \in T_x(S)$, there is $D \in |H|$ such that $x \in D$, but $t \notin T_x(D)$. ■

Theorem 2.11 *Let S be a geometrically ruled surface and let $|H| = |X_0 + \mathfrak{b}f|$ be a base-point-free complete linear system on S. Let $\phi_H : S \longrightarrow \mathbf{P}^N$ be the regular map which $|H|$ defines. Let $K = \{x \in S/x$ is a base point of $|H - Pf|$, for some $P \in X\}$. Then the map ϕ_H is an isomorphism exactly in the open set $S \setminus K$.*

Proof: Let us first see that ϕ_H is injective in $S \setminus K$ and ${\phi_H}^{-1}(\phi_H(S \setminus K)) = S \setminus K$. Given $x \in S/K$ and $y \in S$ with $x \neq y$, they must be separated by elements of $|H|$:

1. Suppose x and y lie in the same generator Pf. As we saw in Proposition 2.3, when $|H|$ is base-point-free, it traces the complete linear system of points of $\mathbf{P}^1$ on each generator Pf. Since this is very ample, it separates x from y, so we can find a divisor D in $|H|$ which meets Pf at x, but not at y.

2. Suppose x and y lie in different generators, $x \in Pf$, $y \in Qf$. Since $x \notin K$, x is not a base point of the linear system $|H - Qf|$. Moreover, the elements of $|H - Qf|$ correspond to the elements of $|H|$ which contain Qf, so we can find a divisor on $|H|$ which contains Q (and $y \in Q$) but not x.

We now check that the differential map $d\phi_H$ is an isomorphism at points $x \in S \setminus K$. In order to get this we will see that $|H|$ separates tangent directions at x, this is, if $t \in T_x(S)$, then there must be an element D in $|H|$ satisfying $x \in D$ but $t \notin T_x(D)$.

1. Suppose $x \in Pf$ and $t \in T_x(Pf)$. Because $|H|$ is base-point-free it traces a very ample system on Pf, so there is an element D in $|H|$ which meets Pf transversally at x and $T_x(D) \neq T_x(Pf)$.

2. Suppose $x \in Pf$ and $t \notin T_x(Pf)$. As $x \notin K$, x is not a base point of $|H - Pf|$. Then there exists a divisor D' in $|H - Pf|$, which doesn't contain x. Taking $D = D' + Pf$, we obtain an element of $|H|$ which contains x and its tangent direction is Pf, so $T_x(D) = T_x(Pf)$ and $t \notin T_x(D)$.

We have seen that ϕ_H is an isomorphism in $S \setminus K$. In fact, we can see that it is not an isomorphism at points of K.

Let $x \in K$ be a point in Pf. Since $x \in K$, x is a base point of the linear system $|H - Qf|$ for some $Q \in X$.

1. If $Q \neq P$, all elements of $|H|$ which contain Qf pass through x, so the image of x by ϕ_H lies at a point of $\phi_H(Qf)$. Thus, there exists $y \in Qf$ with $\phi_H(y) = \phi_H(x)$ and ϕ_H is not bijective in K.

2. Let $Q = P$. Let $C_1 \in |H|$ be a curve which meets Pf transversally at x. It exists because $|H|$ is base-point-free, so $h^0(\mathcal{O}_S(H-Pf)) < h^0(\mathcal{O}_S(H-x))$. Let $t_1 \in T_x(S)$ be the tangent vector to C_1 at x. Suppose that there is other curve $C_2 \in |H|$ which meets Pf transversally at x. Let $t_2 \in T_x(S)$ be its tangent vector at x. Suppose $\langle t_1 \rangle \neq \langle t_2 \rangle$. We can define both curves by local equations u_1 and u_2. Taking $u = \lambda_1 u_1 + \lambda_2 u_2$ we define a curve C on the linear system $|H|$. The tangent vector to C at x is $t = \lambda_1 t_1 + \lambda_2 t_2$. By a suite election of λ_1 and λ_2 we can suppose $t \neq 0$ (so C nonsingular at x) and $\langle t \rangle = T_x(Pf)$, because $\langle t_1 \rangle \neq \langle t_2 \rangle$ and $\langle t_1, t_2 \rangle = T_x(S)$.

   On the other hand, we know that an unisecant irreducible curve can not be tangent to a generator. Then the curve C is on the linear system $|H - Pf|$ and it can be written as $C = Pf + C'$. By hypothesis, x is a base point of $|H - Pf|$, so $x \in C'$. From this, x is a singular point of C and we get a contradiction. Note, that we had supposed that there were two curves on $|H|$ which passed through x with different tangent directions. Then, we deduce that all nonsingular curves at x of $|H|$ have a unique tangent direction $\langle t_1 \rangle$ at x.

   Finally, let us see that $d\phi_H$ is not an isomorphism. In other case, given the tangent vector $t_1 \in T_x(S)$, there must be a curve $D \in |H|$ with $t_1 \notin T_x(D)$. But, if D is nonsingular at x, then $T_x(D) = \langle t_1 \rangle$. If D is singular at x, then $T_x(D) = T_x(S)$ and $t_1 \in T_x(D)$. ■

Remark 2.12 This theorem yields information about the singular locus of a scroll in $\mathbf{P}^N$. Let $R \subset \mathbf{P}^N$ be a linearly normal scroll given by the ruled surface $\mathbf{P}(\mathcal{E})$ and the unisecant complete linear system $|H|$ on $\mathbf{P}(\mathcal{E})$.

As $\mathbf{P}(\mathcal{E})$ is smooth, if ϕ_H is an isomorphism in an open set $U \subset \mathbf{P}(\mathcal{E})$, then R is smooth at points of the image $\phi_H(U)$. The singular locus of R will be supported at points of $R \setminus \phi_H(U)$.

Let us apply Proposition 2.3. Since $|X_0 + \mathfrak{b}f|$ is base-point-free, we have that $h^0(\mathcal{O}_S(X_0 + (\mathfrak{b} - P)f))) = h^0(\mathcal{O}_S(X_0 + \mathfrak{b}f)) - 2$. Furthermore, the linear system $|X_0 + (\mathfrak{b} - P)f|$ has a base point on Qf when $h^0(\mathcal{O}_S(X_0 + (\mathfrak{b} - P - Q)f)) = h^0(\mathcal{O}_S(X_0 + (\mathfrak{b} - P)f)) - 1$ and it has Qf as a fixed component when $h^0(\mathcal{O}_S(X_0 + (\mathfrak{b} - P - Q)f)) = h^0(\mathcal{O}_S(X_0 + (\mathfrak{b} - P)f)$. From this there can appear the following singularities:

1. If $h^0(\mathcal{O}_S(X_0 + (\mathfrak{b} - P - Q)f)) = h^0(\mathcal{O}_S(X_0 + \mathfrak{b}f)) - 3$ with $P \neq Q$, then the generators $\phi_H(Pf)$ and $\phi_H(Qf)$ meet at a unique point.

2. If $h^0(\mathcal{O}_S(X_0 + (\mathfrak{b} - 2P)f)) = h^0(\mathcal{O}_S(X_0 + \mathfrak{b}f)) - 3$, then the generator $\phi_H(Pf)$ meets its infinitely near generator at a unique point. It is called torsal generator.

3. If $h^0(\mathcal{O}_S(X_0 + (\mathfrak{b} - P - Q)f)) = h^0(\mathcal{O}_S(X_0 + \mathfrak{b}f)) - 2$ with $P \neq Q$, then the generators $\phi_H(Pf)$ and $\phi_H(Qf)$ coincide and we have a singular generator.

4. If $h^0(\mathcal{O}_S(X_0 + (\mathfrak{b} - 2P)f)) = h^0(\mathcal{O}_S(X_0 + \mathfrak{b}f)) - 2$, then the generator $\phi_H(Pf)$ coincides with its infinitely near generator and it is again a singular generator.

■

Corollary 2.13 *Let S be a geometrically ruled surface and let $|H| = |X_0 + \mathfrak{b}f|$ be a complete linear system on S. $|H|$ is very ample if and only if $h^0(\mathcal{O}_S(H - (P+Q)f)) = h^0(\mathcal{O}_S(H)) - 4$ for any $P, Q \in X$.*

Proof: $|H|$ is very ample if it is base-point-free and the morphism $\phi_H : S \longrightarrow \mathbf{P}^N$ is an isomorphism.

Let us suppose $|H|$ is very ample. Since it is base-point-free and according to Corollary 2.4, we deduce that $h^0(\mathcal{O}_S(H - Pf)) = h^0(\mathcal{O}_S(H)) - 2$ for any $P \in X$. By the above theorem, as ϕ_H is an isomorphism at any point, $|H - Pf|$ is always base-point-free and $h^0(\mathcal{O}_S(H - (P+Q)f)) = h^0(\mathcal{O}_S(H - Pf)) - 2$ for any $Q \in X$. It follows that:

$$h^0(\mathcal{O}_S(H - (P+Q)f)) = h^0(\mathcal{O}_S(H)) - 4.$$

Conversely, let us suppose $h^0(\mathcal{O}_S(H - (P+Q)f)) = h^0(\mathcal{O}_S(H)) - 4$ for any $P, Q \in X$. If $|H|$ were not base-point-free there would be a point $P \in X$ which satisfies $h^0(\mathcal{O}_S(H - Pf)) \geq h^0(\mathcal{O}_S(H)) - 1$ so $h^0(\mathcal{O}_S(H - (P+Q)f)) \geq h^0(\mathcal{O}_S(H)) - 3$, which contradicts the hypothesis. If ϕ_H were not an isomorphism at a point x, by the above theorem, x would be a base point of $|H - Pf|$ and there would be a $Q \in X$ satisfying $h^0(\mathcal{O}_S(H - (P+Q)f)) \geq h^0(\mathcal{O}_S(H - Pf)) - 1$, so $h^0(\mathcal{O}_S(H - (P+Q)f)) \geq h^0(\mathcal{O}_S(H)) - 3$, which contradicts the hypothesis again. ■

Proposition 2.14 *Let D be a section of a ruled surface S and let $|H| = |D + \mathfrak{b}f|$ be a base-point-free complete linear system. Let $\phi_H : S \longrightarrow \mathbf{P}^N$ the regular map defined by $|D + \mathfrak{b}f|$. If $\mathfrak{b}$ is very ample, then ϕ_H is an isomorphism out of D.*

Proof: Applying Theorem 2.11, we see that it is sufficient to check that $|D + (\mathfrak{b} - P)f|$ is base-point-free out of D.

Let $x \in S \setminus D$ be a point in the generator Qf. Let $P \in X$. Since $\mathfrak{b}$ is very ample, $\mathfrak{b} - P$ is base-point-free and we can take a divisor $\mathfrak{b}' \sim \mathfrak{b} - P$ which does not contain Q. Then, $D + \mathfrak{b}'f \sim D + (\mathfrak{b} - P)f$ does not contain x and this is not a base point of $|D + (\mathfrak{b} - P)f|$. ■

Proposition 2.15 *If $\mathfrak{b}$ and $\mathfrak{b}+\mathfrak{e}$ are very ample divisors on X and $\mathfrak{b}$ is nonspecial, then the complete linear system $|H| = |X_0 + \mathfrak{b}f|$ is very ample.*

Proof: Because $\mathfrak{b}$ is nonspecial and very ample, given $P \in X$, $\mathfrak{b} - P$ is base-point-free and nonspecial.

Applying Lemma 2.1 we see that

$$h^0(\mathcal{O}_S(H - (P+Q)f)) = h^0(\mathcal{O}_X(\mathfrak{b} - P - Q)) + h^0(\mathcal{O}_X(\mathfrak{b} + \mathfrak{e} - P - Q))$$

for any $P, Q \in X$.

Since $\mathfrak{b}$ and $\mathfrak{b}+\mathfrak{e}$ are very ample, $h^0(\mathcal{O}_X(\mathfrak{b}-P-Q)) = h^0(\mathcal{O}_X(\mathfrak{b}))-2$ and $h^0(\mathcal{O}_X(\mathfrak{b}+\mathfrak{e}-P-Q)) = h^0(\mathcal{O}_X(\mathfrak{b}+\mathfrak{e}))-2$; we obtain $h^0(\mathcal{O}_S(H-(P+Q)f) = h^0(\mathcal{O}_S(H)) - 4$ and by Corollary 2.13, $|H|$ is very ample. ■

3 Decomposable ruled surfaces.

Definition 3.1 *Let $S \longrightarrow X$ be a geometrically ruled surface over a nonsingular curve X of genus g. The ruled surface is called decomposable if $\mathcal{E}_0$ is a direct sum of two invertible sheaves.*

The invariant e on a decomposable geometrically ruled surface is positive:

Theorem 3.2 *Let S be a ruled surface over the curve X of genus g, determined by a normalized locally free sheaf $\mathcal{E}_0$.*

1. *If $\mathcal{E}_0$ is decomposable then $\mathcal{E}_0 \cong \mathcal{O}_C \oplus \mathcal{L}$ for some $\mathcal{L}$ with $deg(\mathcal{L}) \leq 0$. Therefore, $e \geq 0$. All values of $e \geq 0$ are possible.*
2. *If $\mathcal{E}_0$ is indecomposable, then $-g \leq e \leq 2g-2$.*

Proof: See [8], V, 2.12. and [15]. ■

Remark 3.3 Geometrically, a decomposable ruled surface has two disjoint unisecant curves. These unisecant curves are given by surjections $\mathcal{E}_0 \cong \mathcal{O}_X \oplus \mathcal{O}_X(\mathfrak{e}) \longrightarrow \mathcal{O}_X(\mathfrak{e}) \longrightarrow 0$ and $\mathcal{E}_0 \cong \mathcal{O}_X \oplus \mathcal{O}_X(\mathfrak{e}) \longrightarrow \mathcal{O}_X \longrightarrow 0$. We denote them by X_0 and X_1. According to ([8], V,2.9), we know $X_1 \sim X_0 - \mathfrak{e}f$.

Since $\mathcal{E}_0$ is decomposable the equality $h^i(\mathcal{O}_S(X_0 + \mathfrak{b}f)) = h^i(\mathcal{O}_X(\mathfrak{b})) + h^i(\mathcal{O}_X(\mathfrak{b}+\mathfrak{e}))$ holds always, because $H^i(\mathcal{O}_S(X_0 + \mathfrak{b})) \cong H^i(\mathcal{E}_0 \otimes \mathcal{O}_X(\mathfrak{b}))$ and $\mathcal{E}_0 \cong \mathcal{O}_X \oplus \mathcal{O}_X(\mathfrak{e})$. ■

Proposition 3.4 *Let S be a decomposable geometrically ruled surface. Let $|H| = |X_0 + \mathfrak{b}f|$ be a complete linear system. Then, $x \in Pf$ is a base point of $|H|$ if and only if it satisfies some of the following conditions:*

1. *P is a base point of $\mathfrak{b}$ and $x = Pf \cap X_1$.*
2. *P is a base point of $\mathfrak{b} + \mathfrak{e}$ and $x = Pf \cap X_0$.*
3. *P is a common base point of $\mathfrak{b}$ and $\mathfrak{b} + \mathfrak{e}$. In this case Pf is fixed component of $|H|$.*

Moreover, $|H|$ is base-point-free if and only if $\mathfrak{b}$ and $\mathfrak{b}+\mathfrak{e}$ are base-point-free.

Proof: Let us examine the trace of the linear system $|X_0 + \mathfrak{b}f|$ on X_0:

$$H^0(\mathcal{O}_S(\mathfrak{b}f)) \hookrightarrow H^0(\mathcal{O}_S(X_0 + \mathfrak{b}f)) \xrightarrow{\alpha} H^0(\mathcal{O}_{X_0}(X_0 + \mathfrak{b}f)) \cong H^0(\mathcal{O}_X(\mathfrak{b} + \mathfrak{e}))$$

According to the above remark, we know that $h^0(\mathcal{O}_S(X_0+\mathfrak{b}f)) = h^0(\mathcal{O}_X(\mathfrak{b})) + h^0(\mathcal{O}_X(\mathfrak{b} + \mathfrak{e}))$, so the map α is a surjection and $|H|$ traces the complete linear system $|\mathfrak{b} + \mathfrak{e}|$ on X_0. Thus, if P is a base point of $\mathfrak{b} + \mathfrak{e}$, then any divisor of $|H|$ meets X_0 at $X_0 \cap Pf$ and conversely.

The same reasoning applies to the trace of $|H|$ on X_1. Since $H^i(\mathcal{O}_{X_1}(X_0 + \mathfrak{b}f)) \cong H^i(\mathcal{O}_X(\mathfrak{b}))$, we can see that P is a base point of $\mathfrak{b}$ if and only if any divisor of $|H|$ meets X_1 at $X_1 \cap Pf$.

Finally, by Remark 3.3, we conclude $h^0(\mathcal{O}_S(H)) - h^0(\mathcal{O}_S(H - Pf)) = (h^0(\mathcal{O}_X(\mathfrak{b})) - h^0(\mathcal{O}_X(\mathfrak{b} - P))) + (h^0(\mathcal{O}_X(\mathfrak{b} + \mathfrak{e})) - h^0(\mathcal{O}_X(\mathfrak{b} + \mathfrak{e} - P)))$. By Proposition 2.5, we see:

1. $|H|$ is base-point-free if and only if $\mathfrak{b}$ and $\mathfrak{b} + \mathfrak{e}$ are base-point-free.
2. $|H|$ has a unique base point on Pf if and only if P is a base point of $\mathfrak{b}$ or $\mathfrak{b} + \mathfrak{e}$, but not both.
3. $|H|$ has Pf as a fixed component if and only if P is a common base point of $\mathfrak{b}$ and $\mathfrak{b} + \mathfrak{e}$. ■

Remark 3.5 The above proof shows us that a complete linear system $|X_0+\mathfrak{b}f|$ traces the complete linear systems $|\mathfrak{b}+\mathfrak{e}|$ and $|\mathfrak{b}|$ on curves X_0 and X_1. Hence, when the linear system $|X_0+\mathfrak{b}f|$ is base-point-free and it defines a regular map on S, X_0 and X_1 apply on linearly normal curves given by the linear systems $|\mathfrak{b} + \mathfrak{e}|$ and $|\mathfrak{b}|$ on X. ■

Theorem 3.6 *Let S be a decomposable ruled surface. The generic element of the complete linear system $|X_0 + \mathfrak{b}f|$ is irreducible if and only if $\mathfrak{b} \sim 0$, $\mathfrak{b} \sim -\mathfrak{e}$ or $\mathfrak{b}$ and $\mathfrak{b} + \mathfrak{e}$ are effective without common base points.*

Proof: Let us first suppose there exists an irreducible element $D \sim X_0 + \mathfrak{b}f$. If $D \sim X_0$, then $\mathfrak{b} \sim 0$ and if $D \sim X_1$, then $\mathfrak{b} \sim -\mathfrak{e}$. In other case, D meets X_0 and X_1 at effective divisors, so $\pi_*(D \cap X_0) \sim \mathfrak{b} + \mathfrak{e}$ and $\pi_*(D \cap X_1) \sim \mathfrak{b}$ must be effective. Furthermore, if $\mathfrak{b}$ and $\mathfrak{b} + \mathfrak{e}$ had a common base point P, then, by Proposition 3.4, Pf would be a fixed component of the linear system and this would not have irreducible elements.

Conversely, if $\mathfrak{b} \sim 0$ or $\mathfrak{b} \sim -\mathfrak{e}$, then $X_0 + \mathfrak{b}f \sim X_0$ or $X_0 + \mathfrak{b}f \sim X_1$ and the generic element is irreducible.

If $\mathfrak{b}$ and $\mathfrak{b} + \mathfrak{e}$ are effective divisors without common base points, the generic point P is not a base point of $\mathfrak{b}$ and $\mathfrak{b} + \mathfrak{e}$, because they are effective. Thus, by Remark 3.3, $h^0(\mathcal{O}_S(X_0 + (\mathfrak{b} - P)f)) = h^0(\mathcal{O}_S(X_0 + \mathfrak{b}f)) - 2$. A finite number of points P can be base points of $\mathfrak{b}$ or $\mathfrak{b} + \mathfrak{e}$ (but not both), so, in this case, $h^0(\mathcal{O}_S(X_0 + (\mathfrak{b} - P)f)) = h^0(\mathcal{O}_S(X_0 + \mathfrak{b}f)) - 1$. Applying Proposition 2.9 the theorem follows. $\blacksquare$

Corollary 3.7 *If $\mathbf{P}(\mathcal{E})$ is a decomposable ruled surface it holds $X_0^2 = -e$, $X_1^2 = e$ and for any other unisecant curve D not linearly equivalent to these, $D^2 \geq e + 2$. In particular:*

1. *If $D \equiv X_0$ then $D \sim X_0$ when $e > 0$ and $D \sim X_0$ or $D \sim X_1$ when $e = 0$. Moreover, if $D \sim X_0$ and $\mathfrak{e} \not\sim 0$, then $D = X_0$.*
2. *If $D \equiv X_1$ then $D \sim X_1$ when $e > 0$ and $D \sim X_0$ or $D \sim X_1$ when $e = 0$. Moreover, if $D \sim X_1$, $e = 0$ and $\mathfrak{e} \not\sim 0$, then $D = X_1$.*

Proof: We know $X_0^2 = -e$ and $X_1^2 = e$. By the above proposition if $D \sim X_0 + \mathfrak{b}f$ is an irreducible curve no linearly equivalent to the first, then $\mathfrak{b}$ and $\mathfrak{b} + \mathfrak{e}$ must be effective divisors, so $deg(\mathfrak{b}) \geq e$ and $deg(\mathfrak{b}) \geq 0$. Since $\mathfrak{b} + \mathfrak{e}$ is effective, if $deg(\mathfrak{b}) = e$, then $\mathfrak{b} \sim -\mathfrak{e}$ and $D \sim X_1$. From this, necessarily $deg(\mathfrak{b}) \geq e + 1$ and $D^2 = 2deg(b) - e \geq e + 2$. $\blacksquare$

Proposition 3.8 *Let S be a decomposable ruled surface. The complete linear system $|X_1| = |X_0 - \mathfrak{e}f|$ satisfies following conditions:*

1. *The set of reducible elements of $|X_1|$ is exactly $\{X_0 + \mathfrak{b}f / \mathfrak{b} \sim -\mathfrak{e}\}$.*
2. *If P is not a base point of $-\mathfrak{e}$, then there exists an irreducible curve of $|X_1|$ passing through any point of Pf not in X_0.*

3. *If P is a base point of $-\mathfrak{e}$, all irreducible curves of $|X_1|$ pass through a unique base point on Pf.*

Proof:

1. Let $D + \mathfrak{b}f$ be a reducible element of $|X_1|$. Since $D + \mathfrak{b}f \sim X_0 - \mathfrak{e}f$, we have $deg(\mathfrak{b}) < deg(-\mathfrak{e})$ and $D \sim X_0 + (-\mathfrak{b} - \mathfrak{e})f$. From this, D is an irreducible curve of self-intersection strictly smaller than X_1. By the above corollary, D must be X_0 and $\mathfrak{b} \sim -\mathfrak{e}$.

2. According to Proposition 3.4, we know that if P is not a base point of $-\mathfrak{e}$, the linear system $|X_1|$ has not base points on Pf. Hence, it traces the complete linear system of points of $\mathbf{P}^1$ on the generator. For each point x of Pf there passes an effective divisor of $|X_1|$ not containing the generator. But, as we see at 1, if $x \notin X_0$, the divisor must be irreducible.

3. According to Proposition 3.4, if P is a base point of $-\mathfrak{e}$, the linear system $|X_1|$ has a base point on the generator Pf, so all irreducible elements of the linear system pass through it. ■

Theorem 3.9 *Let S be a decomposable ruled surface and let $|H| = |X_0 + \mathfrak{b}f|$ be a complete linear system on S. Then:*

1. *If $\mathfrak{b}$ is very ample and $\mathfrak{b} + \mathfrak{e}$ is base-point-free, then $|H|$ defines an isomorphism in $S \setminus X_0$.*

2. *If $\mathfrak{b} + \mathfrak{e}$ is very ample and $\mathfrak{b}$ is base-point-free, then $|H|$ defines an isomorphism in $S \setminus X_1$.*

3. *$|H|$ is very ample if and only if $\mathfrak{b}$ and $\mathfrak{b} + \mathfrak{e}$ are very ample.*

Proof: By Proposition 3.4, if $\mathfrak{b}$ and $\mathfrak{b} + \mathfrak{e}$ are base-point-free the linear system $|X_0 + \mathfrak{b}f|$ is base-point-free.

Since $X_0 + \mathfrak{b}f \sim X_1 + (\mathfrak{b} + \mathfrak{e})f$, we can apply Proposition 2.14. Taking $D = X_0$ or $D = X_1$, we obtain the assertions 1 and 2.

The third equivalence is consequence of Corollary 2.13: $|H|$ is very ample if and only if $h^0(\mathcal{O}_S(H - (P + Q)f)) = h^0(\mathcal{O}_S(H)) - 4$. Now, it is sufficient to remark that in a decomposable ruled surface it holds $h^0(\mathcal{O}_S(H)) = h^0(\mathcal{O}_X(\mathfrak{b}))+ h^0(\mathcal{O}_X(\mathfrak{b} + \mathfrak{e}))$ and $h^0(\mathcal{O}_S(H - (P + Q)f)) = h^0(\mathcal{O}_X(\mathfrak{b} - P - Q)) + h^0(\mathcal{O}_X(\mathfrak{b} + \mathfrak{e} - P - Q))$. ■

Theorem 3.10 *Let S be a decomposable ruled surface and let $|H| = |X_0 + \mathfrak{b}f|$ be a base-point-free linear system. Let $\phi_H : S \longrightarrow R \subset \mathbf{P}^N$ be the map defined by $|H|$. Then:*

1. *$N = h^0(\mathcal{O}_X(\mathfrak{b})) + h^0(\mathcal{O}_X(\mathfrak{b}+\mathfrak{e})) - 1$ and $deg(S) = 2deg(\mathfrak{b}) - e$.*

2. *$\phi_H(X_0)$ and $\phi_H(X_1)$ are linearly normal curves given by the maps $\phi_{\mathfrak{b}+\mathfrak{e}} : X \longrightarrow \phi_H(X_0)$ and $\phi_{\mathfrak{b}} : X \longrightarrow \phi_H(X_1)$. Moreover, they lie in complementary disjoint spaces of $\mathbf{P}^N$.*

3. *The singular locus of R is supported at most in $\phi_H(X_0)$, $\phi_H(X_1)$ and the set $K = \{\phi_H(Pf)/|\mathfrak{b} - P|$ and $|\mathfrak{b} + \mathfrak{e} - P|$ have a common base point$\}$. If $K = S$ the map ϕ_H is not birational. If $K \neq S$ the map ϕ_H is birational and singularities of R are exactly:*

   (a) *Singular unisecant curves $\phi_H(X_0)$ or $\phi_H(X_1)$ if the regular maps $\phi_{\mathfrak{b}+\mathfrak{e}} : X \longrightarrow \phi_H(X_0)$ or $\phi_{\mathfrak{b}} : X \longrightarrow \phi_H(X_1)$ are not birational.*

   (b) *Isolated singularities on $\phi_H(X_0)$ or $\phi_H(X_1)$ when $\mathfrak{b} + \mathfrak{e}$ or $\mathfrak{b}$ are not very ample but they define birational maps. They correspond to the generators Pf and Qf meeting at a point on $\phi_H(X_i)$. If $P = Q$, the generator Pf is a torsal generator.*

   (c) *Double generators when $\mathfrak{b} + \mathfrak{e} - P$ and $\mathfrak{b} - P$ have a common base point Q. Then $\phi_H(Pf) = \phi_H(Qf)$. If $P = Q$, the generator $\phi_H(Qf)$ coincides with its infinitely near generator.*

Proof: The linear system $|H|$ is base-point-free, so it defines a regular map $\phi_H : S \longrightarrow \mathbf{P}^N$. The hyperplane sections of the scroll R correspond to divisors of the linear system $|H|$. If we denote $N_0 = h^0(\mathcal{O}_X(\mathfrak{b} + \mathfrak{e})) - 1$ and $N_1 = h^0(\mathcal{O}_X(\mathfrak{b})) - 1$, then $N = h^0(\mathcal{O}_S(X_0 + \mathfrak{b}f)) - 1 = N_0 + N_1 + 1$ and $deg(R) = deg(H) = (X_0 + \mathfrak{b}f)^2 = 2deg(\mathfrak{b}) - e$.

At Remark 3.5 we saw that curves $\phi_H(X_0)$ and $\phi_H(X_1)$ are linearly normal and they are defined by maps $\phi_{\mathfrak{b}+\mathfrak{e}} : X \longrightarrow \phi_H(X_0) \subset \mathbf{P}(H^0(\mathcal{O}_X(\mathfrak{b} + \mathfrak{e}))^\vee)$ and $\phi_{\mathfrak{b}} : X \longrightarrow \phi_H(X_1) \subset \mathbf{P}(H^0(\mathcal{O}_X(\mathfrak{b}))^\vee)$. Thus $\phi_H(X_0)$ lies in $\mathbf{P}^{N_0}$ and $\phi_H(X_1)$ lies in $\mathbf{P}^{N_1}$. Since $h^0(\mathcal{O}_S(H - X_0 - X_1)) = 0$, there are not hyperplane sections containing both curves. Hence these lie in complementary disjoint spaces.

Finally, let us study the singular locus of R. Applying Theorem 2.11, we know that ϕ_H is an isomorphism out of base points of linear systems $|H - Pf|$, $P \in X$.

As we saw in Proposition 3.4, in a decomposable ruled surface base points of linear system lie in X_0 or X_1, except when there is a base generator. In this case, $\mathfrak{b} - P$ and $\mathfrak{b} + \mathfrak{e} - P$ have a common base point.

It follows that the singular locus of R is supported at most in $\phi_H(X_0)$, $\phi_H(X_1)$ and $K = \{\phi_H(Pf)/|\mathfrak{b} - P|$ and $|\mathfrak{b} + \mathfrak{e} - P|$ have a common base point$\}$. If $K = S$, the map ϕ_H is not an isomorphism at any point so it is not birational. On the contrary, if $K \neq S$ we can see which are exactly the singularities of R.

We will reason on the curve $\phi_H(X_0)$, but similar arguments apply to the curve $\phi_H(X_1)$.

If the morphism $\phi_{\mathfrak{b}+\mathfrak{e}} : X \longrightarrow \phi_H(X_0)$ is not birational, then it is a $k:1$ map and we have an unisecant singular curve on the scroll.

If the morphism $X \longrightarrow \phi_H(X_0)$ is birational, the map given by $\mathfrak{b}+\mathfrak{e}$ is 1:1 in an open set, but isolated singularities can appear. This happens when the divisor $\mathfrak{b}+\mathfrak{e}-P$ has a base point Q for some $P \in X$. Then the linear system $|H-Pf|$ have a base point at $X_0 \cap Qf$:

- If Q is not a base point of $\mathfrak{b}-P$, the linear system $|H-Pf|$ has no more base points in Qf and then the unique singular point in $\phi_H(Pf)$ lies at $\phi_H(X_0) \cap \phi_H(Qf) \cap \phi_H(Pf)$. The generators $\phi_H(Qf)$ and $\phi_H(Pf)$ meet at a point. Moreover, if $Q=P$, the generator $\phi_H(Pf)$ meets its infinitely near generator at a unique point and it is a torsal generator.

- If Q is a base point of $\mathfrak{b}-P$, the linear system $|H-Pf|$ has Qf as a fixed component. Then, both generators coincide in the image, so $\phi_H(Pf)=\phi_H(Qf)$ is a singular generator. If $P=Q$ the generator $\phi_H(Qf)$ coincides with its infinitely near generator. ■

We will finish this section by studying conditions for m-secant divisors to be very ample on a decomposable ruled surface.

We begin with a technical result on computing the dimension of a m-secant linear system on a decomposable ruled surface. It is known that:

$$h^i(\mathcal{O}_S(X_0+\mathfrak{b}f)) = h^i(\mathcal{O}_X(\mathfrak{b})) + h^i(\mathcal{O}_X(\mathfrak{b}+\mathfrak{e})).$$

Let us see the following generalization:

Lemma 3.11 *Let $|mX_0+\mathfrak{b}f|$ be a m-secant linear system on a decomposable ruled surface S. Then,*

$$h^i(\mathcal{O}_S(mX_0+\mathfrak{b}f)) = \sum_{k=0}^{m} h^i(\mathcal{O}_X(\mathfrak{b}+k\mathfrak{e})), \quad i \geq 0$$

Proof: We note that because S is a surface, then $h^i(\mathcal{O}_S(H))=0$ when $i>2$. The proof is by induction on m:

It is clear that $h^i(\mathcal{O}_S(\mathfrak{b}f)) = h^i(\mathcal{O}_X(\mathfrak{b}))$ and in particular $h^2(\mathcal{O}_S(\mathfrak{b}f))=0$.

Assuming the formula holds for $m-1$, we will prove it for m. Let $|H| = |mX_0+\mathfrak{b}f|$ be a m-secant system. Consider the exact sequence:

$$0 \longrightarrow \mathcal{O}_S(H-X_1) \longrightarrow \mathcal{O}_S(H) \longrightarrow \mathcal{O}_{X_1}(H) \longrightarrow 0$$

Since $X_1 \sim X_0 - \mathfrak{e}f$ and introducing cohomology, we have:

$$\begin{array}{ccccccc} 0 & \to & H^0(\mathcal{O}_S(H-X_1)) & \to & H^0(\mathcal{O}_S(H)) & \stackrel{\alpha_0}{\to} & H^0(\mathcal{O}_X(\mathfrak{b})) & \to \\ & \to & H^1(\mathcal{O}_S(H-X_1)) & \to & H^1(\mathcal{O}_S(H)) & \stackrel{\alpha_1}{\to} & H^1(\mathcal{O}_X(\mathfrak{b})) & \to \\ & \to & H^2(\mathcal{O}_S(H-X_1)) & \to & H^2(\mathcal{O}_S(H)) & \to & 0 & \end{array}$$

where $H - X_1 \sim (m-1)X_0 + (\mathfrak{b}+\mathfrak{e})f$. The map α_0 is a surjection because given $\mathfrak{b}' \sim \mathfrak{b}$ we have $\mathfrak{b}' = \alpha(mX_0 + \mathfrak{b}'f)$, where $mX_0 + \mathfrak{b}'f \sim mX_0 + \mathfrak{b}f$. α_1 is a surjection too, because $h^2(\mathcal{O}_S((m-1)X_0 + (\mathfrak{b}+\mathfrak{e})f)) = 0$ by induction hypothesis. We conclude that:

$$\begin{aligned} h^i(\mathcal{O}_S(mX_0 + \mathfrak{b}f)) &= h^i(\mathcal{O}_X(\mathfrak{b})) + \sum_{k=0}^{m-1} h^i(\mathcal{O}_X(\mathfrak{b} + (k+1)\mathfrak{e})) = \\ &= \sum_{k=0}^{m} h^i(\mathcal{O}_X(\mathfrak{b} + k\mathfrak{e})) \end{aligned}$$

■

We will now restrict our attention to study the trace of a m-secant linear system $|mX_0 + \mathfrak{b}f|$ on a generator Pf. Consider the exact sequence:

$$0 \longrightarrow \mathcal{O}_S(mX_0 + (\mathfrak{b} - P)f) \longrightarrow \mathcal{O}_S(mX_0 + \mathfrak{b}f) \longrightarrow \mathcal{O}_{Pf}(mX_0 + \mathfrak{b}f) \longrightarrow 0$$

Introducing cohomology:

$$0 \longrightarrow H^0(\mathcal{O}_S(mX_0 + (\mathfrak{b} - P)f)) \longrightarrow H^0(\mathcal{O}_S(mX_0 + \mathfrak{b}f)) \xrightarrow{\alpha} H^0(\mathcal{O}_{\mathbf{P}^1}(m))$$

We see that $|mX_0 + \mathfrak{b}f|$ traces a linear subsystem of the complete linear system of divisors of degree m on $\mathbf{P}^1$ on the generator Pf.

Let us introduce homogeneous coordinates $[x_0 : x_1]$ on $\mathbf{P}^1$, where point $[0 : 1]$ is $X_0 \cap Pf$ and point $[0 : 1]$ is $X_1 \cap Pf$.

If P is not a base point of $\mathfrak{b}$, then we can choose a divisor $\mathfrak{b}' \sim \mathfrak{b}$ such that $P \notin \mathfrak{b}'$. Hence, $mX_0 + \mathfrak{b}'f \sim mX_0$ traces the point $[1 : 0]$ with multiplicity m on Pf. It is defined by the equation $\{x_0^m = 0\}$.

If P is not a base point of $\mathfrak{b} + m\mathfrak{e}$, then we can choose a divisor $\mathfrak{b}' \sim \mathfrak{b} + m\mathfrak{e}$ such that $P \notin \mathfrak{b}'$. Hence, $mX_1 + \mathfrak{b}'f \sim mX_0$ traces the point $[0 : 1]$ with multiplicity m on Pf. It is defined by the equation $\{x_1^m = 0\}$.

Similarly, if P is not a base point of $\mathfrak{b} + (m-k)\mathfrak{e}$, there exists a divisor $kX_0 + (m-k)X_1 + \mathfrak{b}'f \sim mX_0 + \mathfrak{b}f$ which traces the points defined by the equation $\{x_0^k x_1^{m-k} = 0\}$.

On the other hand, we know that $dim(Im(\alpha)) = h^0(\mathcal{O}_S(mX_0 + \mathfrak{b}f)) - h^0(\mathcal{O}_S(mX_0 + (\mathfrak{b} - P)f))$. According to the lemma above, we have:

$$dim(Im(\alpha)) = \sum_{k=0}^{m} (h^0(\mathcal{O}_X(\mathfrak{b} + k\mathfrak{e})) + h^0(\mathcal{O}_X(\mathfrak{b} - P + k\mathfrak{e})))$$

Since P is a base point of $\mathfrak{b}+k\mathfrak{e}$ if and only if $h^0(\mathcal{O}_X(\mathfrak{b}-P+k\mathfrak{e})) = h^0(\mathcal{O}_X(\mathfrak{b}+k\mathfrak{e}))$, we have found a basis of $Im(\alpha)$:

$$Im(\alpha) = \langle x_0^k x_1^{m-k} / \text{P is not base point of } \mathfrak{b}+(m-k)\mathfrak{e} \rangle$$

Now, we are in a position to examine the base points of the linear system on the generator Pf.

A base point corresponds with a common zero of basis polynomials of $Im(\alpha)$. This zero can be only points $[0:1]$ or $[1:0]$, except when $Im(\alpha) = \{0\}$, equivalently, when P is a base point of $\mathfrak{b}+(m-k)\mathfrak{e}$ for all $k \in \{0,\ldots,m\}$. Thus, we have three possibilities:

1. $[0:1] = X_0 \cap Pf$ is a base point of the linear system. Then, the polynomial $\{x_1^m\}$ cannot be in the basis, so P must be a base point of $\mathfrak{b}+m\mathfrak{e}$.
2. $[1:0] = X_1 \cap Pf$ is a base point of the linear system. Then, the polynomial $\{x_0^m\}$ cannot be in the basis, so P must be a base point of $\mathfrak{b}$.
3. $Im(\alpha) = \{0\}$. In this case, all points of Pf are base points. Moreover, Pf is a fixed component of the linear system and P is a common base point of $\mathfrak{b}+(m-k)\mathfrak{e}$ for all $k \in \{0,\ldots,m\}$.

Hence, we have proved the following proposition:

Proposition 3.12 *Let $|mX_0+\mathfrak{b}f|$ be a m-secant linear system on a decomposable ruled surface. Then $x \in Pf$ is a base point of the linear system if and only if some of the following conditions holds:*

1. *P is a base point of $\mathfrak{b}$ and $x = Pf \cap X_1$.*
2. *P is a base point of $\mathfrak{b}+m\mathfrak{e}$ and $x = Pf \cap X_0$.*
3. *P is a common base point of $\mathfrak{b}+k\mathfrak{e}$, for all $k \in \{0,\ldots,m\}$. In this case Pf is a fixed component of the linear system.*

Furthermore, $|mX_0+\mathfrak{b}f|$ is base-point-free if and only if $\mathfrak{b}$ and $\mathfrak{b}+m\mathfrak{e}$ are base-point-free. ∎

From now on we assume $|mX_0+\mathfrak{b}f|$ to be base-point-free on the generator Pf. Then, the linear system defines a regular map. We are interested in studying when it is an isomorphism.

The above discussion allow us to describe the regular map defined by the linear subsystem $|Im(\alpha)|$:

$$\begin{array}{rcl} \phi : Pf \cong \mathbf{P}^1 & \longrightarrow & \mathbf{P}^m \\ [x_0 : x_1] & \longrightarrow & [\lambda_0 x_0^m : \lambda_1 x_0^{m-1} x_1 : \ldots : \lambda_m x_1^m] \end{array}$$

where $\lambda_i = h^0(\mathcal{O}_X(\mathfrak{b}+(m-i)\mathfrak{e})) - h^0(\mathcal{O}_X(\mathfrak{b}+(m-i)\mathfrak{e}-P))$. Thus, $\lambda_i = 0$ when P is a base point of $\mathfrak{b}+(m-i)\mathfrak{e}$ and $\lambda_i = 1$ in other case.

Because the linear system is base-point-free on Pf, we know that $\lambda_0 = \lambda_m = 1$. Moreover, the map ϕ is an isomorphism when their affine restrictions are isomorphisms:

$$\begin{array}{rcl} \phi_0 : \mathbf{A}^1 & \longrightarrow & \mathbf{A}^m \\ x_1 & \longrightarrow & (\lambda_1 x_1, \lambda_2 x_1^2, \ldots, \lambda_{m-1} x_1^{m-1}, \lambda_m x_1^m) \end{array}$$

$$\begin{array}{rcl} \phi_1 : \mathbf{A}^1 & \longrightarrow & \mathbf{A}^m \\ x_0 & \longrightarrow & (\lambda_0 x_0^m, \lambda_1 x_0^{m-1}, \ldots, \lambda_{m-2} x_0^2, \lambda_{m-1} x_0) \end{array}$$

Each ϕ_i is an isomorphism when it is injective with not null differential at any point. But, $d\phi_0(x_1) = (\lambda_1, 2\lambda_2 x_1, \ldots, (m-1)\lambda_{m-1}x_1^{m-2}, m\lambda_m x_1^{m-1})$, so differential is not null at any point if and only if $\lambda_1 \neq 0$. Moreover, in this case ϕ_0 is injective. The same reasoning applies to ϕ_1. It follows that ϕ is an isomorphism when $\lambda_1 = \lambda_{m-1} = 1$:

Proposition 3.13 *Let $|mX_0 + \mathfrak{b}f|$ be a m-secant linear system on a decomposable ruled surface. The linear system defines an isomorphism on the generator Pf if and only if P is not a base point of $\mathfrak{b}$, $\mathfrak{b}+\mathfrak{e}$, $\mathfrak{b}+(m-1)\mathfrak{e}$ and $\mathfrak{b}+m\mathfrak{e}$.* ■

Finally, we examine when the m-secant linear system defines an isomorphism.

Let us suppose that the complete linear system is base-point-free and the map restricted to the generators is an isomorphism. Then $\mathfrak{b}$, $\mathfrak{b}+\mathfrak{e}$, $\mathfrak{b}+(m-1)\mathfrak{e}$ and $\mathfrak{b}+m\mathfrak{e}$ are base-point-free.

Let us see when the linear system separates points and tangent vectors.

Let x and y be two points on the ruled surface. We study several cases:

1. $x, y \in Pf$. Then, since the restriction of the linear system to the generators defines an isomorphism, the linear system separates points on the same generator.
2. $x \in Qf$, $y \in Pf$, $x \notin X_0$. Then, if $\mathfrak{b}$ is very ample, there is a divisor $\mathfrak{b}' \sim \mathfrak{b}$ which contains Q but not P. It is sufficient to take $mX_0 + \mathfrak{b}'f$ as a divisor on the linear system which contains y but not x.
3. $x \in Qf$, $y \in Pf$, $x \notin X_1$. Similarly, since $mX_0 + \mathfrak{b}f \sim mX_1 + (\mathfrak{b}+m\mathfrak{e})f$, points x and y can be separated if $\mathfrak{b}+m\mathfrak{e}$ is very ample.
4. $x \in Qf$, $y \in Pf$, $x \notin X_0 \cup X_1$. Because $mX_0 + \mathfrak{b}f \sim kX_0 + (m-k)X_1 + (\mathfrak{b}+(m-k)\mathfrak{e})f$, it is sufficient that $\mathfrak{b}+(m-k)\mathfrak{e}$ is very ample for some $k \in \{0, \ldots, m\}$.

Let $x \in Pf$ and $t \in T_x(S)$ be a point and a tangent vector on the surface:

1. If $t \in T_x(Pf)$, then there is a divisor which meets Pf transversally, because the restriction of the linear system to the generators is an isomorphism.

2. If $t \notin T_x(Pf)$ and $x \notin X_0$, then if $\mathfrak{b}$ is very ample we can take a divisor $\mathfrak{b}' \sim \mathfrak{b}$ which passes with multiplicity 1 through P. Thus, $mX_0 + \mathfrak{b}'f$ is a divisor in the linear system which passes through x with the direction of Pf.

3. $t \notin T_x(Pf)$ and $x \notin X_1$, in the same manner, it is sufficient to consider $\mathfrak{b} + m\mathfrak{e}$ very ample.

4. $t \notin T_x(Pf)$ and $x \notin X_0 \cup X_1$, it is sufficient that $\mathfrak{b} + (m-k)\mathfrak{e}$ is very ample for some $k \in \{0, \dots, m\}$.

Therefore we conclude the following theorem:

Theorem 3.14 *Let $|mX_0 + \mathfrak{b}f|$ be a m-secant linear system on a decomposable ruled surface S. It defines a rational map $\phi : S \longrightarrow \mathbf{P}^N$. Then:*

1. *If $\mathfrak{b}$, $\mathfrak{b} + \mathfrak{e}$, $\mathfrak{b} + (m-1)\mathfrak{e}$ and $\mathfrak{b} + m\mathfrak{e}$ are base-point-free and*
   - *(a) $\mathfrak{b} + (m-k)\mathfrak{e}$ is very ample for some $k \in \{0, \dots, m\}$, then ϕ is an isomorphism in $S \setminus (X_0 \cup X_1)$.*
   - *(b) $\mathfrak{b}$ is very ample, then ϕ is an isomorphism in $S \setminus X_0$.*
   - *(c) $\mathfrak{b} + m\mathfrak{e}$ is very ample, then ϕ is an isomorphism in $S \setminus X_1$.*

2. *$|mX_0 + \mathfrak{b}f|$ is very ample if and only if $\mathfrak{b}$, $\mathfrak{b} + \mathfrak{e}$, $\mathfrak{b} + (m-1)\mathfrak{e}$ and $\mathfrak{b} + m\mathfrak{e}$ are base-point-free and $\mathfrak{b}$ and $\mathfrak{b} + m\mathfrak{e}$ are very ample.*

Proof: It remains to prove the necessary conditions for the complete linear system to be very ample.

Let us suppose that $|mX_0 + \mathfrak{b}f|$ is very ample. Then, the generators are applied isomorphically, so $\mathfrak{b}$, $\mathfrak{b} + \mathfrak{e}$, $\mathfrak{b} + (m-1)\mathfrak{e}$ and $\mathfrak{b} + m\mathfrak{e}$ must be base-point-free.

Consider the trace of the linear system on X_0:

$$0 \longrightarrow H^0(\mathcal{O}_S((m-1)X_0 + \mathfrak{b}f)) \longrightarrow H^0(\mathcal{O}_S(mX_0 + \mathfrak{b}f)) \xrightarrow{\alpha} H^0(\mathcal{O}_X(\mathfrak{b} + m\mathfrak{e}))$$

By Lemma 3.11, α is a surjection. The complete linear system $|mX_0 + \mathfrak{b}f|$ traces the complete linear system $|\mathfrak{b} + m\mathfrak{e}|$ on X_0. Since $|mX_0 + \mathfrak{b}f|$ is very

ample, the restriction of the map to X_0 must be an isomorphism, so $|\mathfrak{b} + m\mathfrak{e}|$ is very ample.

Similarly, by examining the trace of the linear system on X_1 we deduce that $\mathfrak{b}$ must be very ample and the conclusion follows. ■

4 Elementary transformation of a ruled surface.

Given a decomposable linearly normal scroll $S \subset \mathbf{P}^N$ of degree d and genus g, if it is projected from a generic nonsingular point we obtain a new linearly normal scroll with the same genus and degree $d-1$. However, curves which were disjoint could intersect in the projection.

This idea will allow us to study an indecomposable scroll as projection from a decomposable one.

In order to work with projections we will use elementary transforms on the abstract model. They are built blowing up the surface at a point and contracting the generator passing through the point.

Let us first remember some properties of the blowing up of a surface and the elementary transformations.

Definition 4.1 *Let S be a smooth surface and let $\epsilon : S_x \longrightarrow S$ be the blowing up of S at $x \in S$. We will denote the exceptional divisor of S_x by $E = \epsilon^{-1}(x)$. Given a curve C in S its strict transform is the curve $\widetilde{C} = \overline{\epsilon^{-1}(C - x)}$.*

Proposition 4.2 *Let $\epsilon : S_x \longrightarrow S$ be the blowing up of S at $x \in S$. Then:*

1. *If C is a curve in S, then $\epsilon^*(C) = \widetilde{C} + \mu_x(C)E$, where $\mu_x(C)$ is the multiplicity of C at x.*
2. *If C is a reduced curve in S_x, then $\epsilon_*(C) = \overline{\epsilon(C \cap (S_x \setminus E))}$.*
3. *If C and D are divisors in S, then $\epsilon^*(C).\epsilon^*(D) = C.D$.*
4. *If C is a divisor in S, then $\epsilon^*(C).E = 0$.*
5. *$E^2 = -1$ (where E is the exceptional divisor).*
6. *If C is a curve in S, then $\widetilde{C}.E = \mu_x(C)$.*
7. *If C and D are curves in S, then $\widetilde{C}.\widetilde{D} = C.D - \mu_x(C)\mu_x(D)$*

8. *If C is a divisor in S and D is a divisor in S_x, then $\epsilon^*(C).D = C.\epsilon_*(D)$.*

Definition 4.3 *Let $\pi : S \longrightarrow X$ be a geometrically ruled surface over a smooth curve X of genus g; let $x \in S$ with $\pi(x) = P$. We will denote the elementary transform of S at x by $S' = elm_x(S)$. It is built blowing up S at x ($\epsilon : S_x \longrightarrow S$) and contracting the exceptional generator $\widetilde{Pf}$ ($\sigma : S_x \longrightarrow S'$.*

The exceptional divisor of the blowing up ϵ is E and it is corresponds to the generator Pf in S'. The exceptional divisor of the contraction σ is $\widetilde{Pf}$.

We will denote the birational map $\sigma^{-1} \circ \epsilon$ by $\nu : S' \longrightarrow S$.

Given a curve C in S we define its strict transform as $C' = \sigma_*(\widetilde{C})$. We will follow denoting the generators of S' as Qf.

If S' is the elementary transform of S at x, then S is the elementary transform of S' at y, where y is the image by σ of the intersection of the exceptional divisors $\widetilde{Pf}$ and E on S_x.

Proposition 4.4 *Let $\pi : S \longrightarrow X$ be a geometrically ruled surface and let $S' = elm_x(S)$ be its elementary transform at point $x \in S$, with $\pi(x) = P$. Then:*

1. *If $\mathfrak{b}$ is a divisor on X, then $\nu^*(\mathfrak{b}f) = \mathfrak{b}f$.*
2. *If C is a n-secant curve on S, then $\nu^*(C) = C' + \mu_x(C)Pf$.*
3. *If C is a n-secant curve on S and D is a m-secant curve on S, then $C'.D' = C.D + nm - n\mu_x(D) - m\mu_x(C)$. Therefore, if C and D are unisecant curves:*
   (a) *If $x \in C \cap D$, then $C'.D' = C.D - 1$.*
   (b) *If $x \notin C \cup D$, then $C'.D' = C.D + 1$.*
   (c) *If $x \in C$, but $x \notin D$, then $C'.D' = C.D$.*
4. *If C is a unisecant curve on S, then $\nu_*\nu*(C) = C + \mathbf{P}f$.*

Proof: We will apply the properties seen in Proposition 4.2:

1. It is sufficient to study how ν^* works on a generator Qf, because any divisor $\mathfrak{b}$ can be written as difference of two effective divisors.

   If $x \in Qf$, then $\nu^*(Qf) = \sigma_*(\epsilon^*(Qf)) = \sigma_*(\widetilde{Qf} + E) = E = Qf$.

   If $x \notin Qf$, then $\nu^*(Qf) = \sigma_*(\epsilon^*(Qf)) = \sigma_*(\widetilde{Qf}) = Qf$.

2. Let C be a n-secant curve on S. Then,

$$\begin{aligned}\nu^*(C) &= \sigma_*(\epsilon^*(C)) = \sigma_*(\widetilde{C} + \mu_x(C)E) = \sigma_*(\widetilde{C}) + \mu_x(C)\sigma_*(E) = \\ &= C' + \mu_x(C).Pf\end{aligned}$$

3. Let C be a n-secant curve and let D be a m-secant curve:

$$C'.D' = \sigma_*(\widetilde{C}).\sigma_*(\widetilde{D}) = \sigma^*(\sigma_*(\widetilde{C})).\widetilde{D}$$

but,

$$\begin{aligned}\sigma^*(\sigma_*(\widetilde{C})) &= \widetilde{\sigma_*(\widetilde{C})} + \mu_x(\sigma_*(\widetilde{C})).\widetilde{Pf} = \widetilde{C} + (\widetilde{\sigma_*(\widetilde{C})}.\widetilde{Pf})\widetilde{Pf} = \\ &= \widetilde{C} + (\widetilde{C}.\widetilde{Pf})\widetilde{Pf} = \widetilde{C} + (C.Pf - \mu_x(Pf)\mu_x(C))\widetilde{Pf} = \\ &= \widetilde{C} + (n - \mu_x(C))\widetilde{Pf}\end{aligned}$$

Then,

$$\begin{aligned}C'.D' &= (\widetilde{C} + (n - \mu_x(C))\widetilde{Pf}).\widetilde{D} = \widetilde{C}.\widetilde{D} + (n - \mu_x(C))(\widetilde{D}.\widetilde{Pf}) = \\ &= C.D - \mu_x(C)\mu_x(D) + (n - \mu_x(C))(m - \mu_x(D)) = \\ &= C.D + nm - n\mu_x(D) - m\mu_x(C)\end{aligned}$$

4. The inverse of ν is the elementary transform of S' at y, where y is the intersection of the two exceptional divisors of S_x. Let C be an unisecant curve:

   (a) If $x \in C$, then $\nu^*(C) = C' + Pf$ and $y \notin C'$, so $\nu_*\nu^* = C'' + Pf = C + Pf$.

   (b) If $x \notin C$, then $\nu^*(C) = C'f$ and $y \in C'$, so $\nu_*\nu^* = C'' + Pf = C + Pf$. ■

Lemma 4.5 *Let $\pi : S \longrightarrow X$ be a geometrically ruled surface and let $\pi' : S' \longrightarrow X$ its elementary transformation at point x, with $x \in Pf$. Let C and D be two unisecant curves on S with $\pi_*(C \cap D) \sim \mathfrak{b}$. Then:*

1. *If $x \in C \cap D$, then $\pi'_*(C' \cap D') \sim \mathfrak{b} - P$.*
2. *If $x \notin C \cup D$, then $\pi'_*(C' \cap D') \sim \mathfrak{b} + P$.*
3. *If $x \in C$ but $x \notin D$, then $\pi'_*(C' \cap D') \sim \mathfrak{b}$.*

Proof: In the above proposition, we saw that if x is at $C \cap D$, then its intersection multiplicity is reduced an unit. In fact, when we blow up at x, both curves are separated at this point so the intersection multiplicity is reduced on Pf.

Similarly, if x is not at $C \cup D$, then its intersection multiplicity grows. In fact, when the generator Pf contracts the new point of intersection is projected on P by π'.

Finally, if $x \in C$, but $x \notin D$, then the intersection of both curves is not modified. ∎

Lemma 4.6 *Let $\pi : S \longrightarrow X$ be a geometrically ruled surface, let $\pi' : S' \longrightarrow X$ be the elementary transformation of S at a point $x \in Pf$. Let C be an unisecant irreducible curve on S with $\mathcal{O}_C(C) \cong \mathcal{O}_X(\mathfrak{b})$ where $\mathfrak{b}$ is a divisor on X. Then $\mathcal{O}_{C'}(C') \cong \mathcal{O}_X(\mathfrak{b} + (1 - 2\mu_x(C))P)$.*

Proof: Let us consider two irreducible curves D_1 and D_2 in S which are linearly equivalents to $C + \mathfrak{a}f$. They exist if we take $\mathfrak{a}$ of degree high; in this way, by Proposition 2.15, the linear system $|C + \mathfrak{a}|$ is very ample and the curves D_i correspond to two hyperplane sections not passing through x.

We know that $\pi_*(D_1 \cap D_2) \sim \mathfrak{b} + 2\mathfrak{a}$ and by Lemma 4.5 $\pi'_*(D'_1 \cap D'_2) \sim \mathfrak{b} + 2\mathfrak{a} + P$. Furthermore, $D_i \sim C + \mathfrak{a}f \Rightarrow D'_i \sim C' + \mu_x(C)Pf + \mathfrak{a}f$, so we obtain that $\pi'_*(D'_1 \cap D'_2) \sim \mathfrak{b}' + 2(\mathfrak{a} + \mu_x(C))$, where $\mathfrak{b}'$ verifies $\mathcal{O}_{C'}(C') \cong \mathcal{O}_X(\mathfrak{b}')$.

Comparing the two expressions we conclude that $\mathfrak{b}' \sim \mathfrak{b} + (1 - 2\mu_x(C))P$. ∎

Let us see how the elementary transform of the abstract model of a scroll corresponds to the projection of the scroll from a point. This is the classical geometrical meaning of the elementary transform introduced by Nagata in [14].

Let R be a scroll in $\mathbf{P}^N$ and let $\phi_H : \mathbf{P}(\mathcal{E}) \longrightarrow R$ be the map induced by a complete unisecant linear system $|H|$ on the ruled surface $\pi : \mathbf{P}(\mathcal{E}) \longrightarrow X$. Let $y = \phi_H(x)$ be a point of R, with $\pi(x) = P$. We will project from y obtaining a scroll R' in $\mathbf{P}^{N-1}$.

Projecting R from y corresponds to take the hyperplane sections of R passing through y. Hence, we are considering elements of the linear system $|H|$ which contain x, that is, the linear subsystem $|H - x|$.

We are interested in obtaining R' as the image of a ruled surface $\mathbf{P}(\mathcal{E}')$ by the map corresponding to a complete unisecant linear system. In order to get this, we make the elementary transformation ν of $\mathbf{P}(\mathcal{E})$ at x. We will denote it by $\mathbf{P}(\mathcal{E}')$. Let us see that the linear system $|H - x|$ in $\mathbf{P}(\mathcal{E})$ corresponds to the complete linear system $|\nu^*(H) - Pf|$ in $\mathbf{P}(\mathcal{E}')$:

Let C be a curve on a unisecant linear system $|D|$. We know that $\nu^*(C) \sim \nu^*(D)$.

If $x \notin C$, then $\nu^*(C) = C'$ and we have $C' \sim \nu^*(D)$.

If $x \in C$, then, since C is unisecant, $\nu^*(C) = C' + Pf$ and we have $C' \sim \nu^*(D) - Pf$.

From this, the elements of the complete linear system $|\nu^*(H) - Pf|$ come from elements of $|H - Pf|$ which don't pass through x or elements of $|H|$ which pass through x. Since $|H - Pf| \subset |H - x|$, the divisors of the linear system $|\nu^*(H) - Pf|$ are strictly the divisors of the linear system $|H - x|$.

Suppose that $y = \phi_H(x)$ is a nonsingular point. As we see at the proof of Theorem 2.11, we know that the linear system $|H|$ separates x from any other point. Hence, the linear subsystem $|H - x|$ is base-point-free, so the corresponding system $|\nu^*(H) - Pf|$ is base-point-free too and it defines a regular map.

The projected scroll R' is given by the morphism defined by the complete linear system $|\nu^*(H) - Pf|$ on the ruled surface $\mathbf{P}(\mathcal{E}')$:

$$\begin{tikzcd}
\mathbf{P}(\mathcal{E}) \arrow[rr, "|H|"] & & R \subset \mathbf{P}^N \arrow[d] \\
\mathbf{P}(\mathcal{E}') \arrow[u, "\nu"] \arrow[rr, "|\nu^*(H) - Pf|"] & & R' \subset \mathbf{P}^{N-1}
\end{tikzcd}$$

The degree of R is H^2 and the degree of R' is $(\nu^*(H) - Pf)^2 = \nu^*(H)^2 - 2$. Applying elementary transform's properties we conclude $\nu^*(H)^2 = (H' + \mu_x(H)Pf)^2 = H^2 + 1$. Therefore, the degree of R' is exactly $H^2 - 1$, that is, the degree of R is reduced in an unit.

Note that at the above discussion we have actually proved the following lemma:

Lemma 4.7 *Let S be a geometrically ruled surface and let $\nu : S' \longrightarrow S$ be its elementary transform at point $x \in S$, $\pi(x) = P$. Let C be an m-secant irreducible curve on S and $\mathfrak{b}$ a divisor on X. Then,*

$$|\nu^*(C) + \mathfrak{a}f| \cong |C + (\mathfrak{a} + mP)f - mx|$$

In fact, if C is an unisecant curve:

1. *If x is not a base point of $|C + (\mathfrak{b} + P)f|$, then $h^0(\mathcal{O}_{S'}(\nu^*(C) + \mathfrak{b}f)) = h^0(\mathcal{O}_S(C + (\mathfrak{b} + P)f)) - 1$.*
2. *If x is a base point of $|C + (\mathfrak{b} + P)f|$, then $h^0(\mathcal{O}_{S'}(\nu^*(C) + \mathfrak{b}f)) = h^0(\mathcal{O}_S(C + (\mathfrak{b} + P)f))$.* ■

Remark 4.8 Let us return to the equivalence between scrolls and locally free sheaves of rank 2. Let $\mathcal{E}_1 \cong \pi_* \mathcal{O}_{P(\mathcal{E})}(H)$ be the locally free sheaf of rank 2 corresponding to the scroll R.

When we project from $x \in \mathbf{P}(\mathcal{E})$, we are taking the sections which pass through x (the sections of the sheaf vanishing at x). Thus, if we consider the exact sequence:

$$0 \longrightarrow \mathcal{E}_2 \longrightarrow \mathcal{E}_1 \longrightarrow \mathcal{O}_{\pi(x)} \longrightarrow 0$$

the kernel $\mathcal{E}_2$ is the locally free sheaf of rank 2 corresponding to the scroll which is the projection of R from $y = \phi_H(x)$. Therefore, if $\nu : \mathbf{P}(\mathcal{E}') \longrightarrow \mathbf{P}(\mathcal{E})$ is the elementary transformation of $\mathbf{P}(\mathcal{E})$ at x, we have that:

$$\mathcal{E}_2 \cong \pi'_* \mathcal{O}_{P(\mathcal{E}')}(\nu^*(H) - Pf)$$

and then $\mathbf{P}(\mathcal{E}_2) \cong \mathbf{P}(\mathcal{E}')$.

Conversely, from the properties of the elementary transformation, we deduce that, if $\mathcal{E}_2 \cong \pi'_* \mathcal{O}_{P(\mathcal{E}')}(D)$ then $\mathcal{E}_1 \cong \pi_* \mathcal{O}_{P(\mathcal{E}')}(\nu_*(D))$ ∎

We will now study explicitly how a ruled surface is modified by applying an elementary transformation. We are interested at the variation of invariant e and the curve of minimum self-intersection. Moreover, we will see when the obtained ruled surface is decomposable.

Theorem 4.9 *Let $\pi : \mathbf{P}(\mathcal{E}_0) \longrightarrow X$ be a ruled surface. Let $x \in X_0$ be a point in the minimum self-intersection curve, with $\pi(x) = P$. Let S' be the elementary transform of S at x. Then, S' is a ruled surface corresponding to a normalized sheaf $\mathcal{E}'_0$ with $\bigwedge^2 \mathcal{E}'_0 \cong \mathcal{O}_X(\mathfrak{e}')$ satisfying $\mathfrak{e}' \sim \mathfrak{e} - P$ ($e' = e + 1$). Furthermore, the minimum self-intersection curve of S' is X'_0.*

Proof: We know that the minimum self-intersection curve in S is X_0 and it satisfies $X_0^2 = -e$. Any other unisecant curve D of S satisfies $D^2 \geq -e$.

According to the elementary transform's properties seen in Proposition 4.4, we see that if $x \in D$, then $D'^2 = D^2 - 1$ and if $x \notin D$, then $D'^2 = D^2 + 1$. From this, since $x \in X_0$, $X_0'^2 = X_0^2 - 1$ and for any other unisecant curve $D'^2 \geq D^2 - 1 \geq X_0^2 - 1 = X_0'^2$.

It follows that X'_0 is the minimum self-intersection curve of S'. Moreover, $e' = -X_0'^2 = -X_0^2 + 1 = e + 1$. By Lemma 4.6, $\mathcal{O}_{X'_0}(X'_0) \cong \mathcal{O}_X(\mathfrak{e} - P)$ and then $\mathfrak{e}' \sim \mathfrak{e} - P$. ∎

Corollary 4.10 *Any indecomposable ruled surface is obtained from a decomposable one applying a finite number of elementary transformations.*

Proof: Let S_0 be an indecomposable ruled surface with invariant e_0. By the theorem above, if we apply an elementary transformation to S_0 at a point in the minimum self-intersection curve, then we obtain a new ruled surface S_1 with invariant $e_1 = e_0 + 1$. S_0 is obtained from S_1 applying the inverse of an elementary transformation which is an elementary transformation too.

We continue in this fashion obtaining a ruled surface S_n with invariant $e_n = e_0 + n$. By Theorem (3.2), the invariant e of an indecomposable ruled surface satisfies $e \leq 2g - 2$. Hence, taking n large enough, $e_n > 2g - 2$ and then S_n is decomposable. ∎

Note that from the above results, Nagata Theorem ([21],V,§1) can be obtained directly:

Corollary 4.11 *Any ruled surface over the curve X is obtained from $X \times \mathbf{P}^1$ applying a finite number of elementary transformations.*

Proof: Let S be a ruled surface over the curve X. By Corollary 4.10, applying a finite number of elementary transformations to S, we can obtain a decomposable ruled surface $S_0 \cong \mathbf{P}(\mathcal{O}_X \oplus \mathcal{O}_X(\mathfrak{e}_0))$, where $-\mathfrak{e}_0$ is an effective divisor.

On the other hand, the ruled surface $X \times \mathbf{P}^1$ corresponds to the decomposable model $\mathbf{P}(\mathcal{O}_X \oplus \mathcal{O}_X)$. We can make elementary transformations on the curve of minimum self-intersection X_0 at the points $P_i f \cap X_0$ with $-\mathfrak{e}_0 \sim P_1 + \ldots + P_e$. By Theorem 4.9 we get the ruled surface S_0. Since S is obtained from S_0 applying a finite number of elementary transformations, the conclusion follows. ∎

Theorem 4.12 *Let $\pi : S \longrightarrow X$ be a decomposable geometrically ruled surface. Let $x \in$ with $\pi(x) = P$. Let S' be the elementary transform of S at x corresponding to a normalized sheaf $\mathcal{E}'_0$ with invariant $e' = -deg(\mathfrak{e}') =$. Let Y_0 be the minimum self-intersection curve of S'. Then:*

1. *If $x \in X_0$, then S' is decomposable, $\mathfrak{e}' \sim \mathfrak{e} - P$ $(e' = e + 1)$ and $Y_0 = X'_0$.*
2. *If $x \in X_1$, then S' is decomposable. Moreover, if $e \geq 1$, then $\mathfrak{e}' \sim \mathfrak{e} + P$ $(e' = e - 1)$ and $Y_0 = X'_0$; if $e = 0$, then $\mathfrak{e}' \sim -\mathfrak{e} - P$ $(e' = e + 1)$ and $Y_0 = X'_1$.*
3. *If $h^0(\mathcal{O}_X(-\mathfrak{e})) > 0$, P is not base point of $-\mathfrak{e}$ and $x \notin X_0$, then S' is decomposable. Moreover, if $e \geq 1$, then $\mathfrak{e}' \sim \mathfrak{e} + P$ $(e' = e - 1)$ and $Y_0 = X'_0$; if $e = 0$, then $\mathfrak{e}' \sim -\mathfrak{e} - P$ $(e' = e + 1)$ and $Y_0 = X'_1$.*
4. *If $x \notin X_0$, $x \notin X_1$ and P is base point of $-\mathfrak{e}$, then S' is indecomposable, $\mathfrak{e}' \sim \mathfrak{e} + P$ $(e' = e - 1)$ and $Y_0 = X'_0$.*

Proof:

1. We saw in Theorem 4.9 that the elementary transform at a point in X_0 satisfies $\mathfrak{e}' \sim \mathfrak{e} - P$ ($e' = e + 1$) and $Y_0 = X_0'$. It remains to prove that S' is decomposable.

   We know that $X_0.X_1 = 0$ in S. Applying the properties of elementary transform we see that $X_0'.X_1' = X_0.X_1 = 0$, because $x \in X_0$ and $x \notin X_1$. Hence, there are two disjoint unisecant curves in S' and this is decomposable.

2. If $x \in X_1$, then $x \notin X_0$ so, by the argument above, we have $X_0'.X_1' = X_0.X_1 = 0$ and S' is a decomposable ruled surface.

   We have $X_0^2 = -e$, $X_1^2 = e$ and for any other unisecant curve D, $D^2 \geq e$. When we apply an elementary transformation at a point of X_1, we obtain $X_0^2 = -e + 1$, $X_1^2 = e - 1$ and $D'^2 \geq e - 1$.

   From this, if $e \geq 1$, then $-e + 1 \leq e - 1$ and X_0' is the minimum self-intersection curve. By Lemma 4.6 and since $x \notin X_0$, we obtain that $\mathcal{O}_{X_0'}(X_0') \cong \mathcal{O}_X(\mathfrak{e} + P)$ and then $\mathfrak{e}' \sim \mathfrak{e} + P$.

   If $e = 0$, then $-e+1 > e-1$ and then X_1' is the minimum self-intersection curve. Applying Lemma 4.6 we obtain that $\mathcal{O}_{X_1'}(X_1') \cong \mathcal{O}_X(-\mathfrak{e} - P)$ and $\mathfrak{e}' \sim -\mathfrak{e} - P$.

3. By Proposition 3.8, if $h^0(\mathcal{O}_X(-\mathfrak{e})) > 0$ then there is an irreducible curve $D \sim X_1$ which passes through any point of S, except through the point in the generator Pf, with P base point of $-\mathfrak{e}$.

   In this case P is not a base point of $-\mathfrak{e}$, so we can take X_1 on the linear system $|X_0 - \mathfrak{e}f|$ passing through x and then we can apply 2.

4. If $x \notin X_0$, $x \notin X_1$ and P is a base point of $-\mathfrak{e}$ then we know that there is not any irreducible curve on $|X_1|$ which passes through x. Hence, by Corollary 3.7, any curve which passes through x satisfies $D^2 \geq e + 2$.

   In this way, we have $X_0'^2 = X_0^2 + 1 = -e + 1$, $X_1'^2 = X_1^2 + 1 = e + 1$ and for any other unisecant curve D, if $x \in D$ then $D'^2 = D^2 - 1 \geq e + 1$ and if $x \notin D$ then $D'^2 = D^2 + 1 \geq e + 3$.

   Since S is decomposable, $e \geq 0$ and $-e + 1 \leq e + 1$, so X_0' is the minimum self-intersection curve of S' and by Lemma 4.6 $\mathfrak{e}' \sim \mathfrak{e} + P$.

   Finally, let us see that S' is indecomposable. We will see that any two unisecant curves intersect. We know $C'.D' = (C'^2 + D'^2)/2$. Moreover, X_0' is the unique curve with self-intersection $-e + 1$ and other curve satisfies $D'^2 \geq e + 1$. Then, $C'.D' \geq (-e + 1 + e + 1)/2 > 0$. ∎

5 Speciality of a scroll.

Definition 5.1 *Let $R \subset \mathbf{P}^N$ be a linearly normal scroll. Let $S = \mathbf{P}(\mathcal{E}_0)$ and H the associated ruled surface and linear system. We call the specialty of R to the superabundance of the linear system $|H|$, that is, $i(R) := h^1(\mathcal{O}_S(H))$. Since H is* 1*-secant, $\pi_* \mathcal{O}_S(H) \cong \mathcal{E}$ is a locally free sheaf of rank* 2 *and $H^i(S, \mathcal{O}_S(H)) \cong H^i(X, \mathcal{E})$.*

Let us interpret the speciality of R according to Riemann-Roch Theorem. Let $W \subset \mathbf{P}^N$ a linear subspace and let us consider the projection $\pi_W : R-(W \cap R) \longrightarrow R' \subset \mathbf{P}^{N'}$. The rational map $\phi' := \pi_W \circ \phi_H : S \longrightarrow R' \subset \mathbf{P}^{N'}$ corresponds to the linear subsystem $\delta := \{H \in |H| : \phi_H^*(W \cap R) \subset Supp(H)\} \subset |H|$, which is defined by the base points $\phi_H^*(W \cap R)$. Hence, ϕ' is not regular strictly at $A = \phi_H^*(W \cap R)$.

We can solve the indeterminations of ϕ' by using the blowing up $\sigma : \widetilde{S}_A \longrightarrow S$ at A ([8],II, example 7.17.3). There is a one-to-one correspondence between the linear system δ and the complete linear system $|\pi * (H) - E_A|$ ([8], V, §4), where $E_A = \sum_{P \in A} E_P$, E_P is the exceptional divisor of the blowing up of S at P and the map $\widetilde{S} \xrightarrow{\sigma} S \xrightarrow{\phi'} \mathbf{R}' \subset \mathbf{P}^{N'}$ is regular at every point. We see that R' is the birational image of a smooth surface $\widetilde{S}$ which is not a ruled surface. But, E_P and $\widetilde{Pf}$ are exceptional on $\widetilde{S}$ for any $P \in A$. By Castelnuovo Theorem we can consider each $\widetilde{Pf}$ as the exceptional divisor of the blowing up of a geometrically ruled surface S^t which is the elementary transform of S at the points of A.

Let us work with the ruled surface $S = \mathbf{P}(\mathcal{E}_0)$ and the 1-secant complete linear system $|H|$. If $A \subset S$ is a set of points, then we have the linear subsystem

$$\delta = \delta_A := \{H \in |H| : A \subset Supp(H)\} \cong \mathbf{P}(H^0(\mathcal{O}_S(H) \otimes \mathcal{I}_A)) \subset |H|$$

It is clear that $\delta_A = \mathbf{P}(H^0(\mathcal{O}_S(H) \otimes \mathcal{I}_A))$ where $\mathcal{I}_A$ is the ideal sheaf of A and $\mathcal{O}_S(H) \otimes \mathcal{I}_A$ is not an invertible sheaf.

Let us consider the blowing up of S at A, $\sigma : \widetilde{S_A} \longrightarrow S$. δ corresponds to the complete linear system $|\sigma^*(H) - E_A| = \mathbf{P}(H^0(\mathcal{O}_{\widetilde{S}}(\sigma^* H) \otimes \mathcal{O}_{\widetilde{S}}(-E_A)))$ (Hartshorne, [8], V, section 4).

Let $D \in \delta$; $D \sim H$ and $\sigma^* D = \widetilde{D} + E_A \sim \sigma^* H$. Then $\widetilde{D} \sim \sigma^* H - E_A$. We see that the strict transform $\widetilde{\delta}$ of δ is a complete linear system which can be considered as a linear subsystem of $|\sigma^* H|$ by adding the exceptional divisor E_A:

$$\begin{array}{rccl} +E_A : & \widetilde{\delta} & \longrightarrow & |\sigma^* H| \\ & \widetilde{\delta} & \longrightarrow & \widetilde{D} + E_A = \sigma^* D \end{array}$$

Thus, the complete linear system corresponds to the sections of the invertible sheaf $\mathcal{O}_{\widetilde{S}}(\sigma^* H) \otimes \mathcal{O}_{\widetilde{S}}(-E_A) \cong \sigma^*(\mathcal{O}_S(H) \otimes \mathcal{I}_A)$.

Moreover, let $\overline{D} \subset \widetilde{S}$ be a divisor and let us consider the restriction map $\sigma|_{\overline{D}} : \overline{D} \longrightarrow \sigma(\overline{D}) = D$. Let $\overline{D} = \sum \overline{D_i}$; then, $\sigma_*(\overline{D_i}) = 0 \iff \overline{D_i}$ is exceptional for σ ([9], 12.20) and $\sigma_* \overline{D_i} = \partial(\sigma|_{\overline{D_i}})\sigma(\overline{D_i})$ in other case.

From this, $\overline{D} \supset E_A \iff \sigma_* \overline{D} \supset A$, and we have the following isomorphism:

$$\sigma_*(\mathcal{O}_{\widetilde{S}}(\sigma^* H) \otimes \mathcal{O}_{\widetilde{S}}(-E_A)) \cong \mathcal{O}_S(H) \otimes \mathcal{I}_A$$

In particular, $\mathcal{O}_S(H) \otimes \mathcal{I}_A$ and $\mathcal{O}_{\widetilde{S}}(\sigma^* H) \otimes \mathcal{O}_{\widetilde{S}}(-E_A)$ define the same linear systems $|\sigma^* H - E_A| \cong \delta$.

In fact, let us see that they have the same cohomology. In order to get this, we will use the following result: given $\sigma : \widetilde{S} \longrightarrow S$ and $\mathcal{F}$ a coherent sheaf in $\widetilde{S}$ such that $\mathcal{R}^i \sigma_* \mathcal{F} = 0$ for all $i > 0$, then $H^i(\widetilde{S}, \mathcal{F}) \cong H^i(S, \sigma_* \mathcal{F})$ for all $i \geq 0$ (Hartshorne [8], chap. III, ex. 8.1).

We apply this to compute:

$$(\mathcal{R}^i \sigma_* \mathcal{O}_{\widetilde{S}}(\sigma^* H - E_A))_x \cong H^i(\sigma^{-1}(x), \mathcal{O}_{\widetilde{S}}(\sigma^* H - E_A)|_{\sigma^{-1}(x)})$$

If $x \notin A$, then $\sigma^{-1}(x)$ is a point and $H^i(\sigma^{-1}(x), \mathcal{O}_{\widetilde{S}}(\sigma^* H - E_A)) = 0$ for all $i > 0$. If $x \in A$, then $\sigma^{-1}(x) = E_x$; $H^i(E_x, \mathcal{O}_{\widetilde{S}}(\sigma^* H - E_A)|_x) = 0$ for all $i > 1$. If $i = 1$, then we have that $E_x^2 = -1$; $E_x . E_y = 0$ for $x \neq y$ and $\sigma^* H . E_x = 0$ for any H, so $H^1(\mathbf{P}^1, \mathcal{O}_{\mathbf{P}^1}(1)) = 0$.

Finally, given $\widetilde{\phi} : \widetilde{S} \xrightarrow{\sigma} S \xrightarrow{\phi'} R' \subset \mathbf{P}^{N'}$ which is defined by the complete linear system $\widetilde{\delta}$ we know that:

$$i(R') = h^1(\mathcal{O}_{\widetilde{S}}(\sigma^* H - E_A)) = h^1(\mathcal{O}_S(H) \otimes \mathcal{I}_A)$$

Let us consider the exact sequence:

$$0 \longrightarrow \mathcal{O}_S(H) \otimes \mathcal{I}_A \longrightarrow \mathcal{O}_S(H) \longrightarrow \mathcal{O}_A \longrightarrow 0 \tag{1}$$

and let us take the long exact sequence of cohomology:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & H^0(\mathcal{O}_S(H) \otimes \mathcal{I}_A) & \longrightarrow & H^0(\mathcal{O}_S(H)) & \xrightarrow{\nu} & H^0(\mathcal{O}_A) & \longrightarrow \\ & \longrightarrow & H^1(\mathcal{O}_S(H) \otimes \mathcal{I}_A) & \longrightarrow & H^1(\mathcal{O}_S(H)) & \longrightarrow & 0 & \end{array}$$

where, if $A = \sum N_x . x$, $H^0(\mathcal{O}_A) \cong \bigoplus_{x \in Supp(A)} \mathbf{C}^{n_x}$, $Im(\nu) = \bigoplus_{\substack{y \in Supp(A) \\ y \text{ assigned to } \delta}} \mathbf{C}^{n_y}$.

We find that:

$$i(R') - i(R) = h^1(\mathcal{I}_A(H)) - h^1(\mathcal{O}_S(H)) = deg(\sum_{\substack{z \in Supp(A) \\ z \text{ not assigned to } \delta}} n_z . z)$$

Referring to $W \cap R$ such that $A = \phi_H^*(W \cap R)$, we obtain that:

$$i(R') - i(R) = \partial(W \cap R) - (dim(\langle W \cap R \rangle) + 1)$$

This discussion gives us the geometrical meaning of the speciality of a scroll according to Riemann-Roch Theorem. The speciality grows exactly the degree of the cycle consisting of the unassigned points of the linear subsystem.

Remark 5.2 We consider S' the elementary transform of S at A. The exact sequence similar to (1) in S' is:

$$0 \longrightarrow \mathcal{O}_{S'}(\sigma^* H) \otimes \mathcal{O}_{S'}(-E_A) \longrightarrow \mathcal{O}_{S'}(\sigma^* H) \longrightarrow \mathcal{O}_{E_A} \longrightarrow 0$$

and it gives the exact sequence:

$$\begin{array}{rcl} 0 & \rightarrow & \pi_*(\mathcal{O}_{S'}(\sigma^* H) \otimes \mathcal{O}_{S'}(-E_A)) \rightarrow \pi_* \mathcal{O}_{S'}(\sigma^* H) \rightarrow \pi_* \mathcal{O}_{E_A} \rightarrow \\ & \rightarrow & \mathcal{R}^1 \pi_*(\mathcal{O}_{S'}(\sigma^* H) \otimes \mathcal{O}_{S'}(-E_A)) \cong 0 \end{array}$$

We have $\pi_* \mathcal{O}_{S'}(\sigma^* H) \cong \pi_* \mathcal{O}_S(H) \cong \mathcal{E}$, $\pi_*(\mathcal{O}_{E_A}) \cong \mathcal{O}_{\pi(A)}$ and $\pi_*(\mathcal{O}_{S'}(\sigma^* H) \otimes \mathcal{O}_{S'}(-E_A)) \cong \mathcal{E}'$. So the exact sequence

$$0 \longrightarrow \mathcal{E}' \longrightarrow \mathcal{E} \longrightarrow \mathcal{O}_{\pi(A)} \longrightarrow 0$$

is the sequence that Maruyama considers in [12] to define the locally free sheaf $\mathcal{E}'$ as the elementary transform of $\mathcal{E}$ at the points $\pi(A)$. ■

Remark 5.3 Nagata Theorem asserts that, if X is a smooth curve of genus $g > 0$, any geometrically ruled surface $\pi : \mathbf{P}(\mathcal{E}) \longrightarrow X$ is a minimal model. Furthermore $\mathbf{P}(\mathcal{E})$ is obtained from $X \times \mathbf{P}^1$ by applying a finite number of elementary transformations.

Recall that $X \times \mathbf{P}^1 \cong \mathbf{P}(\mathcal{E}_0)$ and $\mathcal{E}_0 \cong \mathcal{O}_X \otimes \mathcal{O}_X$ so $h^1(\mathcal{E}_0) = 2h^1(\mathcal{O}_X) = 2h^0(\mathcal{K}) = 2g$. Moreover, if $\mathcal{E}$ is other locally free sheaf of rank 2 such that $\mathbf{P}(\mathcal{E}_0) \cong \mathbf{P}(\mathcal{E})$, then $\mathcal{E} \cong \mathcal{E}_0 \otimes \mathcal{O}_X(\mathfrak{a})$ with $\mathfrak{a} \in Pic(X)$. Hence, $h^1(\mathcal{E}) = 2h^1(\mathcal{O}_X(\mathfrak{a}))$ is always even. We can obtain a projective model of $X \times \mathbf{P}^1$ which is a scroll with speciality 2, but never with speciality 1. This interpretation poses to study the existence of geometrically ruled surfaces $\pi : \mathbf{P}(\mathcal{E}) \longrightarrow X$ and unisecant linear systems H with $h^1(\mathcal{O}_S(H)) = 1$ defining scrolls of speciality 1, such that any special scroll is obtained by projection from them. We will call them canonical surfaces because, analogous to canonical curves, they provide the geometrical meaning of speciality according to Riemann-Roch Theorem. In fact, at [5], we will characterize these surfaces as decomposable ruled surfaces that contain a canonical curve in the minimal 2-secant divisor class. ■

6 Segre Theorems.

In this section we review the results of Segre about special scrolls that appeared in [18].

The first simple example of special scroll is a cone. Let X be a smooth curve of genus $g \geq 1$ and $|\mathfrak{b}|$ a base-point-free linear system on X defining a birational map $\phi_{\mathfrak{b}} : X \longrightarrow P^{N-1}$. Let $\overline{X}$ be the linearly normal curve image of X by $\phi_{\mathfrak{b}}$. Let $R \subset \mathbf{P}^N$ be the cone over $\overline{X}$.

It is clear that R is a linearly normal scroll, because the base curve $\overline{X}$ is linearly normal. In particular, R is the image of the decomposable ruled surface $S = \mathbf{P}(\mathcal{O}_X \oplus \mathcal{O}_X(-\mathfrak{b}))$ by the complete linear system $|X_1| = |X_0 + \mathfrak{b}f|$. The curve X_0 of S goes into the vertex of the cone, but the curve X_1 corresponds to its hyperplane section. The speciality of the cone R is:

$$i(R) = h^1(\mathcal{O}_S(X_1)) = h^1(\mathcal{O}_X) + h^1(\mathcal{O}_X(\mathfrak{b})) = g + h^1(\mathcal{O}_X(\mathfrak{b})) > 0$$

Note that in this case the generic hyperplane section of the scroll R is a linearly normal curve. Segre proved that this fact characterizes the cones over non rational curves:

Theorem 6.1 *Let $R \subset \mathbf{P}^N$ be a linearly normal scroll of genus $g \geq 1$, with $N \geq 3$. Then R is a cone if and only if the generic hyperplane section is linearly normal.*

Proof: It remains to prove that R is a cone when the generic hyperplane section is linearly normal. Suppose that R is a scroll verifying this condition.

Let S be the ruled surface over the curve X of genus g and $|H|$ the complete linear system associated to R.

Consider the exact sequence:

$$0 \longrightarrow H^0(\mathcal{O}_S) \longrightarrow H^0(\mathcal{O}_S(H)) \longrightarrow H^0(\mathcal{O}_H(H))$$

where $\mathcal{O}_H(H) \cong \mathcal{O}_X(\mathfrak{b})$ and $\mathfrak{b}$ is the divisor of X corresponding to the hyperplane section of R.

Because the hyperplane section of H is linearly normal, the sequence is exact on the right, and then:

$$h^0(\mathcal{O}_S(H)) = h^0(\mathcal{O}_X(\mathfrak{b})) + 1$$

Suppose that there is an unisecant irreducible curve D on S, $D \sim H - Pf$, with $P \in X$. We take the exact sequence:

$$0 \longrightarrow H^0(\mathcal{O}_S(Pf)) \longrightarrow H^0(\mathcal{O}_S(H)) \longrightarrow H^0(\mathcal{O}_D(H))$$

Since R is not rational, $h^0(\mathcal{O}_S(Pf)) = h^0(\mathcal{O}_X(P)) = 1$. Moreover $\mathcal{O}_D(H) \cong \mathcal{O}_X(\mathfrak{b} - P)$. Thus:

$$h^0(\mathcal{O}_S(H)) \leq 1 + h^0(\mathcal{O}_X(\mathfrak{b} - P))$$

We obtain that $h^0(\mathcal{O}_X(\mathfrak{b})) \leq h^0(\mathcal{O}_X(\mathfrak{b} - P))$ and then $\mathfrak{b}$ has a base point. But this is no possible, because the linear system $|H|$ is base-point-free.

We deduce that there is not any irreducible unisecant curve D on S, such that $D \sim H - Pf$ for any $P \in X$. By Theorem 2.7, this is equivalent to:

$$h^0(\mathcal{O}_S(H - Pf - Qf)) \geq h^0(\mathcal{O}_S(H)) - 3 \text{ for all } P, Q \in X$$

According to Remark 2.12 this means that any pair of generators of R intersect. But, given a family of lines such that any pair of them intersects, either all the lines are on a plane, or all the lines pass through a point. Since $N > 2$ it follows that R is a cone. ■

We mean a directrix curve in the scroll as the image of a section of the ruled surface. It seems probably that a special scroll has a directrix curve. C. Segre proved it giving a condition over the degree of the scroll. In fact, this is always true (see [5]). However, here we rescue the results of Segre.

Let $R \subset \mathbf{P}^N$ be a scroll defined by the unisecant complete linear system $|H|$ on the geometrically ruled surface $\pi : \mathbf{P}(\mathcal{E}_0) = S \longrightarrow X$. We will denote its speciality by $i = h^1(\mathcal{O}_S(H))$ and its degree by $deg(R) = h^2 = d$. By Riemann-Roch Theorem we know that $N = d - 2g + 1 + i$.

Geometrically, in order to find directrix curves on the scroll R we consider hyperplane sections passing through some generators. Thus, we work in S with the linear subsystems $|H - \mathfrak{a}f| \subset |H|$ where $\mathfrak{a}$ is an effective divisor on X of degree a.

The linear system $|H - \mathfrak{a}f|$ is not empty when $h^0(\mathcal{O}_S(H - \mathfrak{a}f)) \geq 1$. Since $h^0(\mathcal{O}_S(H-Pf)) \geq h^0(\mathcal{O}_S(H))-2$, we see that $h^0(\mathcal{O}_S(H-\mathfrak{a}f)) \geq h^0(\mathcal{O}_S(H)) - 2a$ and there will be elements in $|H - \mathfrak{a}f|$ if

$$2a \leq h^0(\mathcal{O}_S(H)) - 1 \iff 2a \leq d - 2g + 1 + i$$

An element of the linear subsystem $|H - \mathfrak{a}f|$ is composed by an unisecant irreducible curve C and a set of generators $\mathfrak{b}f$ with $\mathfrak{b} \in Pic(X)$ and $b = \partial(\mathfrak{b})$. Let us consider the exact sequence:

$$0 \longrightarrow \mathcal{O}_S(H - C) \longrightarrow \mathcal{O}_S(H) \longrightarrow \mathcal{O}_C(H) \longrightarrow 0$$

Since $C + \mathfrak{b}f \in |H - \mathfrak{a}f|$, $H - C \sim (\mathfrak{a} + \mathfrak{b})f$. Thus, applying cohomology we obtain:

$$\begin{array}{cccccccc} 0 & \longrightarrow & H^0(\mathcal{O}_X(\mathfrak{a}+\mathfrak{b})) & \longrightarrow & H^0(\mathcal{O}_S(H)) & \xrightarrow{\alpha} & H^0(\mathcal{O}_C(H)) & \longrightarrow \\ & \longrightarrow & H^1(\mathcal{O}_X(\mathfrak{a}+\mathfrak{b})) & \longrightarrow & H^1(\mathcal{O}_S(H)) & \longrightarrow & H^1(\mathcal{O}_C(H)) & \longrightarrow & 0 \end{array} \tag{2}$$

$h^1(\mathcal{O}_C(H))$ is the speciality of the curve C in the scroll R and $h^1(\mathcal{O}_S(H))$ is the speciality of the scroll. Hence, from the exact sequence we get a first result:

Proposition 6.2 *The speciality of a directrix curve of a scroll is less than or equal to the speciality of the scroll.* ■

Let us suppose that R is special: $i \geq 1$. By Riemann-Roch Theorem:

$$h^0(\mathcal{O}_C(H)) = H.C - g + 1 + h^1(\mathcal{O}_C(H))$$

where $deg(C) = H.C = H.(H - (\mathfrak{a} + \mathfrak{b})f) = d - (a + b)$. Furthermore,

$$\begin{array}{rcl} dim(Im(\alpha)) & = & h^0(\mathcal{O}_S(H)) - h^0(\mathcal{O}_X(\mathfrak{a} + \mathfrak{b})) \quad = \\ & = & d - 2g + 2 + i - h^0(\mathcal{O}_X(\mathfrak{a} + \mathfrak{b})) \quad \geq \\ & \geq & d - 2g + 2 + i - (h^0(\mathcal{O}_X(\mathfrak{a})) + b) \end{array}$$

But, $dim(Im(\alpha)) \leq h^0(\mathcal{O}_C(H))$, so we obtain:

$$h^1(\mathcal{O}_C(H)) \geq -h^0(\mathcal{O}_X(\mathfrak{a})) + a + i - g + 1$$

By the semicontinuity of the cohomology, a generic effective divisor $\mathfrak{a}$ on a curve X of genus g verify:

$$\begin{array}{ll} h^0(\mathcal{O}_X(\mathfrak{a})) = a - g + 1 & \text{if } a \geq g \\ h^0(\mathcal{O}_X(\mathfrak{a})) = 1 & \text{if } a \leq g \end{array}$$

In particular, if $\mathfrak{a}$ consists of $g - i + 1$ generic points, since $i \geq 1$, $a \leq g$ and we have that:

$$h^1(\mathcal{O}_C(H)) \geq -h^0(\mathcal{O}_X(\mathfrak{a})) + a + i - g + 1 \geq -1 + g - i + 1 + i - g + 1 = 1,$$

and then the curve C is special.

We have only required that the system $|H - \mathfrak{a}f|$ is not empty when $a = g - i + 1$. This happens when $2(g - i + 1) \leq d - 2g + 1 + i$. Since $i \geq 1$, it is sufficient that $d \geq 4g - 2$:

Proposition 6.3 *A special scroll R of genus g and degree $d \geq 4g - 2$ has a special directrix curve.* ■

From now on we assume that $d \geq 4g - 2$. C is an special curve, so $deg(C) \leq 2g - 2$. For any other curve D we have that $deg(D) + deg(C) \geq d \geq 4g - 2$. Therefore, $deg(D) \geq 2g > deg(C)$ and C is the curve of minimum degree of the scroll and, equivalently, the curve of minimum self-intersection of S. Moreover, C is the unique special curve of R.

Finally, $deg(C) = d - (a+b) \leq 2g-2$, so $a + b \geq d - (2g-2) \geq 2g$. Thus, $\mathfrak{a} + \mathfrak{b}$ is a nonspecial divisor ($h^1(\mathcal{O}_X(\mathfrak{a} + \mathfrak{b})) = 0$). From the exact sequence (2) we deduce that C is linearly normal (because α is an surjection) and that C and R have the same speciality: $h^1(\mathcal{O}_C(H)) = h^1(\mathcal{O}_S(H))$. We conclude the following theorem:

Theorem 6.4 *A special scroll R of genus g and degree $d \geq 4g-2$ have a unique special directrix curve C. Moreover, C is the curve of minimum degree of the scroll, it is linearly normal and it has its same speciality.* ■

Finally we remark that C.Segre proved a result which relates the speciality of a scroll and the speciality of a proper bisecant curve. A bisecant curve on the scroll is proper when it has not double points out of the singular locus of R. The Theorem motivates the construction of the canonical scrolls made at [5], that is, scrolls playing a similar role to the canonical curves.

Theorem 6.5 *A proper bisecant curve on a linearly normal scroll R is linearly normal and it has its same speciality.*

Proof: See [5]. ■

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